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A Nonlinear Dynamical Model of the Interaction between Soil Degradation, Agricultural Productivity and Food Security: Well-posedness, Threshold Dynamics, Hopf Bifurcation and Management Scenarios

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Abstract

We develop and analyse a four-dimensional nonlinear ordinary differential equation model for normalised soil quality S, agricultural productivity A, food security F and soil-degradation pressure D, closed by the hypothesis that low food security raises production pressure and hence degradation. We prove positivity, boundedness and global existence on an explicit invariant region. The equilibrium problem is reduced to one strictly monotone scalar equation, which gives a sharp threshold: a unique positive equilibrium E* exists if and only if the productivity number RA=ra++/ exceeds one, and when RA1 the productivity-free equilibrium is globally asymptotically stable. At E* we derive the Jacobian and characteristic polynomial, prove that all four coefficients are positive for every admissible parameter set, and conclude that E* can lose stability only through a Hopf bifurcation, with an explicit Routh–Hurwitz criterion. A centre-manifold calculation shows that the transcritical bifurcation at RA=1 is forward. Numerical work, with hypothetical parameter values, gives eigenvalues and eigenvectors for four management scenarios, local and global sensitivity rankings, and a verified supercritical Hopf bifurcation in a stress-test parameter set. In the illustrative baseline, combining soil restoration with reduced degradation intensity raises equilibrium soil quality by about 110%, productivity by about 60% and food security by about 27%, with sub-additive gains. Global stability of E* is not claimed, and the results are not calibrated to data.

Keywords: Soil Degradation, Food Security, Nonlinear Dynamical System, Routh–Hurwitz Criterion, Transcritical Bifurcation, Hopf Bifurcation, Sensitivity Analysis

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1. INTRODUCTION

Crop production rests on the condition of the soil. Soil is the medium for root development and the reservoir from which crops draw water and nutrients, so a decline in soil condition reduces productive capacity and, through production, the availability and stability of food. Soil quality reflects interacting physical, chemical and biological properties and the soil's capacity to support food production and environmental functions [3]. Degradation is correspondingly multidimensional: it includes erosion, nutrient depletion, loss of organic matter, structural deterioration, salinisation, acidification, compaction and decline in biological activity. The Food and Agriculture Organization defines land degradation as a long-term decline in the ability of land to deliver essential ecosystem functions and services, and estimates that about 1.7 billion people live where human-induced degradation has lowered crop yields by at least 10% [7].

Management acts on soil in both directions. Fertiliser is necessary where soils are nutrient-poor, but inappropriate or excessive use can acidify soil and harm soil biology, so that productivity gains are bought at the cost of soil quality [5,22]. Conservation agriculture, organic amendments and residue retention can improve soil condition; in a long-term field experiment, conservation agriculture raised an index of soil health and sustained crop yields under experimental warming [24]. The productivity response is heterogeneous: a systematic review and meta-analysis of agroecological practices in Africa found that effects on land productivity depend on the practice, crop, climate and soil properties [23]. Food security has four dimensions, namely availability, access, utilisation and stability [18], so production is a major driver of food security but not its sole determinant.

The causal arrows also return. Where food security is low there may be pressure to expand or intensify production, and intensification without sustainable management can degrade soil further, which lowers productivity and food security again. Whether and how strongly this feedback operates depends on prices, technology, tenure and policy; it is a modelling hypothesis, not an established law, and is treated as such here. Because soil, productivity and food security change over time, interact, and respond to interventions with delays, a static relationship among them cannot show whether the system settles, collapses or oscillates, nor where the thresholds lie. A small system of nonlinear ordinary differential equations can answer these questions by proof as well as by simulation.

Mathematical models of agricultural dynamics have been analysed with compartment models of agricultural, industrial and ecospheric assets, using approximate series solutions and analysis of the uniqueness and stability of equilibria [1], and with household-level models linking assets, nutrients, water and soil quality [22]. The contribution of this paper is a compact model in which the complete loop from soil quality to productivity, food security, degradation pressure and back to soil quality is analysable in closed form. We prove well-posedness; derive a sharp existence threshold and a global extinction result below it; obtain a complete local stability criterion with a Hopf-only route to instability; establish the direction of the transcritical bifurcation; and quantify sensitivity and intervention effects numerically. The aim is to identify the mathematical mechanisms linking soil degradation, productivity and food security under different management scenarios. No claim is made that the model reproduces any real system: parameters are hypothetical.

The paper is organised as follows. Section 2 reviews related work and states the research gap. Section 3 formulates the model and proves well-posedness. Section 4 contains the equilibrium, stability and bifurcation analysis. Section 5 reports numerical results and sensitivity analysis. Section 6 discusses the findings and limitations, and Section 7 concludes with recommendations.

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2. RELATED WORK AND RESEARCH GAP

- **Soil degradation, soil quality and yield:** Land degradation reduces crop yields at large scale [7], and soil quality is a composite of physical, chemical and biological attributes [3]. Field evidence links soil management to soil health and yield [24], but responses are heterogeneous across practices, crops, climates and soils [23]. These findings motivate interaction terms, such as the product of soil quality and productivity growth used here, rather than a fixed linear soil–yield law.
- **Food security:** Agricultural systems models have mostly tracked availability, and Nicholson et al. [18] show with dynamic models how access indicators and stability metrics can be added. The present model uses a composite index F with a saturating gain from productivity and a decay rate, so access, utilisation and stability are represented implicitly, not resolved.
- **Dynamical models of agroecosystems:** Radosavljevic et al. [22] build nonlinear ODE models coupling household assets, phosphorus, water and soil quality, with logistic regeneration of soil quality, and study attractors and basins of attraction to compare poverty-alleviation sequences; their results include that input-led intensification can reinforce poverty by degrading soil quality. Abanum et al. [1] analyse a four-compartment model of agricultural, auxiliary agricultural, industrial and ecospheric assets using the Laplace decomposition method, with uniqueness and equilibrium-stability analysis. That model describes asset or biomass compartments; as far as its abstract indicates, it does not contain a soil-quality or food-security index or feedback from food insecurity to production pressure.
- **Analytical tools:** We use linearisation, centre-manifold reduction and bifurcation theory [21,14,26], the Routh–Hurwitz criterion [8], Hopf bifurcation criteria [9,15], threshold numbers in the spirit of the next-generation approach [25], and local and global sensitivity analysis [4,16,17,26].
- **Research gap:** Within the sources reviewed, (i) soil-science studies measure yield responses while food-security modelling focuses on availability and access, so the two are rarely coupled in one dynamical system; (ii) existing coupled models emphasise household or asset dynamics and numerically computed attractors, not closed-form threshold numbers and complete local-stability criteria; (iii) a food-insecurity-to-production-pressure feedback is rarely made an explicit state-space loop whose stability can be proved; and (iv) analytical treatment of loss of stability through Hopf bifurcation in such loops was not found. This is a statement about the sources reviewed, not a claim that no such work exists.

3. MODEL FORMULATION

3.1 Assumptions and variables

The state variables are the soil-quality index $S \in [0,1]$, the productivity index $A \in [0,1]$, the food-security index $F \in [0,1]$ and the degradation-pressure index $D \geq 0$, all dimensionless and normalised. The assumptions are:

(H1) Soil regenerates by self-reinforcing processes (rate r_s) and by external restoration effort (rate $\rho > 0$), both towards full quality $S = 1$.

(H2) Soil quality is lost through agricultural activity (αA) and through accumulated degradation pressure (χD). Degradation pressure never improves soil.

(H3) Per-capita productivity growth rises with soil quality (βS) and, by hypothesis, with food insecurity ($\xi(1 - F)$, the production-pressure feedback). Growth is limited by a shared capacity $A = 1$, reduced by

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degradation (γD), and offset by a loss rate θ (abandonment, pests, labour and input constraints).

(H4) Productivity raises food security with saturation ($\eta A(1 - F)$); food security decays at rate μ.

(H5) Degradation pressure is generated by agricultural activity, more strongly when food security is low ($\kappa(1 - F)$), and dissipates at rate δ.

(H6) Parameters are constant, the system is deterministic, and external shocks (climate, prices, policy) are excluded from the baseline.

3.2 The model

$$\begin{aligned}
\dot{S} &= (1 - S)\left(r_s S + \rho\right) - S(\alpha A + \chi D),\\
\dot{A} &= A[B(S,F)(1 - A) - \gamma D - \theta],\\
B(S,F) &= r_a + \beta S + \xi(1 - F),\\
\dot{F} &= \eta A(1 - F) - \mu F,\\
\dot{D} &= \lambda A\,[1 + \kappa(1 - F)] - \delta D
\end{aligned} \tag{1}$$

with all parameters positive. Put $\Lambda = r_a + \beta + \xi$ and $D_{max} = \lambda(1 + \kappa)/\delta$. The terms are interpreted as follows:

In the soil equation, $r_s S(1 - S)$ is self-reinforcing regeneration, $\rho(1 - S)$ is external restoration proportional to the soil deficit, $-\alpha AS$ is activity-driven loss and $-\chi DS$ is loss caused by accumulated degradation pressure. In the productivity equation, $A\left(r_a + \beta S + \xi(1 - F)\right)(1 - A)$ is capacity-limited growth with a baseline, a soil benefit and a production-pressure term, $-\gamma DA$ is the degradation penalty and $-\theta A$ is productivity loss. In the food-security equation, $\eta A(1 - F)$ is the saturating gain from productivity and $-\mu F$ is decay. In the degradation equation, $\lambda A[1 + \kappa(1 - F)]$ is pressure generated by activity and amplified under food insecurity, and $-\delta D$ is dissipation.

Three design choices deserve comment. First, the growth rate of A is factored through $(1 - A)$ and the loss rate θ is explicit: with separate logistic and linear growth terms, $A = 1$ can give a positive growth rate and A can leave $[0, 1]$, and without θ the productivity-free state is always unstable, so no threshold exists. Second, soil loss is written as $(1 - S)\left(r_s S + \rho\right) - S(\alpha A + \chi D)$, which makes $S \leq 1$ transparent and ensures that degradation pressure never improves soil. Third, D is kept as a state variable because it carries memory: it responds to activity with a lag set by δ, and this lag is what allows oscillation. Eliminating D by quasi-steady state removes the oscillatory regime.

Table 1 lists the parameters. All baseline values are hypothetical: they were chosen to give an interior equilibrium with a stable, interpretable baseline and are not literature-derived or calibrated.

Table 1. Model parameters. Baseline values are hypothetical.

Symbol	Meaning (units yr^{-1} unless stated)	Sign	Baseline
r_s	soil self-regeneration	+	0.30
ρ	restoration effort	+	0.05

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Symbol	Meaning (units yr^{-1} unless stated)	Sign	Baseline
α	activity-driven soil loss	+	0.20
χ	soil loss caused by degradation pressure	+	0.40
r_a	baseline productivity growth	+	0.50
β	soil-quality benefit to productivity	+	0.60
γ	degradation penalty on productivity	+	0.50
ξ	insecurity-driven production pressure	+	0.20
θ	productivity loss rate	+	0.30
η	food-security gain from productivity	+	0.80
μ	food-security decay	+	0.30
λ	degradation pressure generated per unit activity	+	0.50
κ	amplification under food insecurity (dimensionless)	+	1.0
δ	pressure dissipation	+	0.40

3.3 Well-posedness

Theorem 1 (Positivity, Invariance, Boundedness). The set $\Omega = [0,1]^3 \times \left[0, D_{max}\right]$ is positively invariant under (1). Solutions starting in Ω exist for all $t \geq 0$ and are unique. For initial data with $S, A, F \in [0,1]$ and any $D(0) \geq 0$, solutions stay bounded and $limsup_{t\to\infty} D(t) \leq D_{max}$.

Proof. The right-hand side is polynomial, hence locally Lipschitz, so a unique local solution exists [9]. On the faces of Ω:

At $S = 0, \dot{S} = \rho > 0$; at $S = 1, \dot{S} = -(\alpha A + \chi D) \leq 0$;

At $A = 0, \dot{A} = 0$ (the plane is invariant); at $A = 1, \dot{A} = -\gamma D - \theta < 0$;

At $F = 0, \dot{F} = \eta A \geq 0$; at $F = 1, \dot{F} = -\mu < 0$;

At $D = 0, \dot{D} = \lambda A[1 + \kappa(1 - F)] \geq 0$; and at $D = D_{max}$,

$\dot{D} = \lambda A[1 + \kappa(1 - F)] - \lambda(1 + \kappa) \leq 0$ because $A \leq 1$ and $1 - F \leq 1$.

The field never points out of Ω, so Ω is positively invariant; in particular nonnegative data give nonnegative solutions. For $D(0) > D_{max}$, the variables S, A, F remain in $[0,1]$ and $\dot{D} \leq \lambda(1 + \kappa) - \delta D$, so $limsup D \leq D_{max}$. On Ω the Jacobian is bounded, so the field is globally Lipschitz there, giving uniqueness and continuous dependence. ▫

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4. MATHEMATICAL ANALYSIS

4.1 Equilibrium and the threshold number

Define

$$R_A = \frac{r_a+\beta+\xi}{\theta} = \frac{\Lambda}{\theta}, \qquad m(A) = \frac{\mu}{\eta A+\mu},$$
$$H(A) = A\,[1 + \kappa m(A)], \qquad G(A) = \alpha A + \chi\frac{\lambda}{\delta}H(A). \tag{2}$$

R_A is the maximal per-capita productivity growth rate divided by the productivity loss rate. Note that $H'(A) = 1 + \kappa\mu^2/(\eta A + \mu)^2 > 0$, so G is strictly increasing.

Theorem 2 (Equilibria). (a) $E_0 = (1, 0, 0, 0)$ always exists and is the only equilibrium with $A = 0$. (b) An equilibrium with $A > 0$ exists if and only if $R_A > 1$, and then it is unique:

$$F^* = \frac{\eta A^*}{(\eta A^*+\mu)}, \; D^* = \frac{\lambda}{\delta}H(A^*),$$
$$S^* = \frac{\left(r_s-\rho-G^*\right)+\sqrt{\left(r_s-\rho-G^*\right)^2+4r_s\rho}}{2r_s}, \tag{3}$$

Where $G^* = G(A^*)$, $A^* \in (0, 1)$ is the unique root of

$$\Phi(A) = \left[r_a + \beta S(A) + \xi m(A)\right](1 - A) - \theta - \gamma\frac{\lambda}{\delta}H(A), \tag{4}$$

and $S(A)$ is the positive root of $r_s S^2 + \left(\rho + G(A) - r_s\right)S - \rho = 0$. Moreover $E^* \in int\Omega$.

Proof. (a) If $A = 0$, then $\dot{F} = -\mu F = 0$ gives $F = 0$, then $D = 0$, and $\dot{S} = (1 - S)\left(r_s S + \rho\right) = 0$ gives $S = 1$ because $r_s S + \rho > 0$.

(b) Let $A > 0$. Setting $\dot{F} = \dot{D} = 0$ gives F^* and D^*, with $1 - F = m(A)$. Setting $\dot{S} = 0$ gives $p(S) := r_s S^2 + \left(\rho + G - r_s\right)S - \rho = 0$. Since $p(0) = -\rho < 0$, $p(1) = G > 0$ and p is an upward parabola, it has exactly one positive root $S(A) \in (0, 1)$, where $p'(S) > 0$ and $\partial p/\partial G = S > 0$; hence $S'(A) = -S\,G'(A)/p'(S) < 0$, so $S(A)$ is strictly decreasing, with $S(0) = 1$.

Setting $\dot{A} = 0$ with $A > 0$ requires $\Phi(A) = 0$. The factor $B(A) = r_a + \beta S(A) + \xi m(A) > 0$ is non-increasing and $1 - A$ is strictly decreasing and positive on $[0, 1)$, so $B(A)(1 - A)$ is strictly decreasing; $-\gamma(\lambda/\delta)H(A)$ is strictly decreasing. Hence Φ is strictly decreasing on [0, 1], with $\Phi(0) = \Lambda - \theta$ and $\Phi(1) = -\theta - \gamma(\lambda/\delta)H(1) < 0$. A root in $(0, 1)$ exists if and only if $\Lambda > \theta$, i.e. $R_A > 1$, and it is unique. Finally, $F^* \in (0, 1)$ and $0 < D^* < \left(\frac{\lambda}{\delta}\right)(1 + \kappa) = D_{max}$ since $A^* < 1$. ▫

Corollary 1 (Structure). E^* depends on (η, μ) only through η/μ and on (λ, δ) only through λ/δ. The interior equilibrium is unique, so the model is monostable and *multiple interior equilibria cannot occur*. Bistability would require an extension (for example a nonlinear soil-to-productivity response).

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Proposition 1 (Bounds). If $R_A > 1$ then $0 < A^* < 1 - 1/R_A$ and $S^* \geq \rho/(\rho + G^*) > 0$.

Proof . At A^*, $B(1 - A^*) = \theta + \gamma D^* > \theta$ and $B \leq \Lambda$, so $1 - A^* > \theta/\Lambda = 1/R_A$. For S^*, $p(S^*) = 0$ gives $S^*(\rho + G^*) = \rho + r_s S^*(1 - S^*) \geq \rho$.

$E_0 = (1, 0, 0, 0)$ is a productivity-free state with full soil quality: without cultivation there is no degradation pressure and soil recovers. A separate severely degraded equilibrium does not exist, because restoration $\rho > 0$ keeps $S^* > 0$ (Proposition 1); severe degradation appears instead as a small S^* in the unique interior equilibrium when degradation intensity is high (Table 4, "unsustainable practice": $S^* = 0.077$). At the baseline $R_A = 4.333$ and

$$E^* = (S^*, A^*, F^*, D^*) + (0.3530,\ 0.2911,\ 0.4370,\ 0.5688),$$

with residuals of order 10^{-17} on substitution into (1). Figure 1 illustrates Theorem 2.

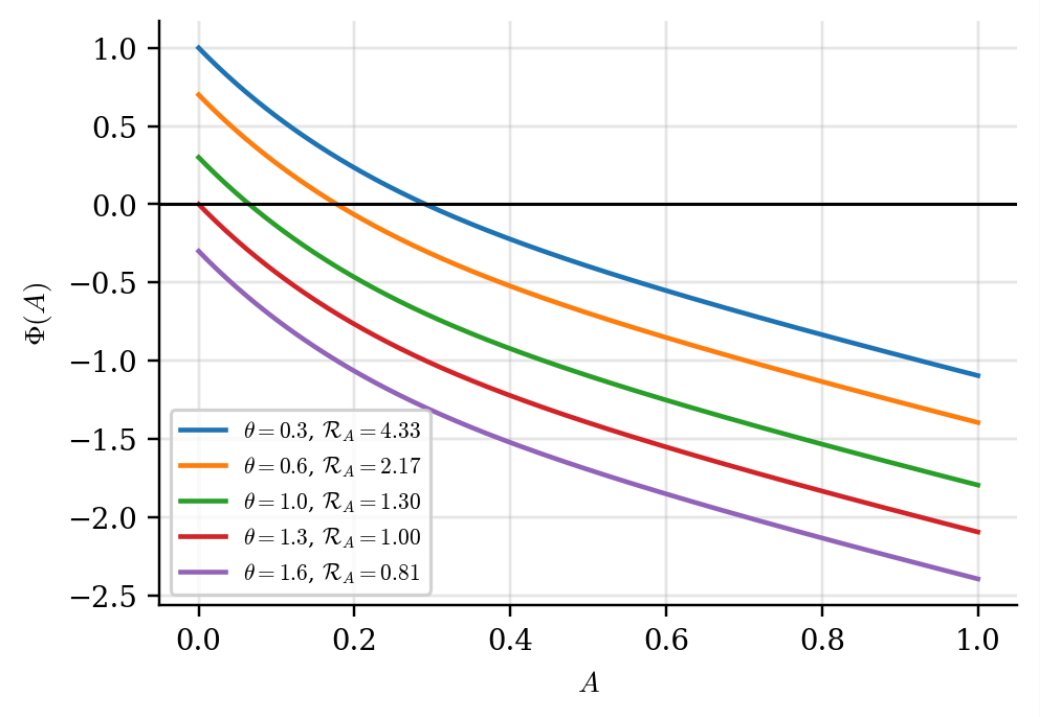


Figure 1. The function $\Phi(A)$ ***for several values of θ. Φ is strictly decreasing and has a root in*** $(0, 1)$ ***exactly when*** $R_A > 1$.

4.2 Jacobian and characteristic polynomial

Write $\Gamma = B(1 - A) - \gamma D - \theta$, the bracket in the productivity equation. Differentiating each equation gives

$$J = \begin{pmatrix} r_s(1-2S) - \rho - \alpha A - \chi D & -\alpha S & 0 & -\chi S \\ A\beta(1-A) & \Gamma - AB & -A\xi(1-A) & -\gamma A \\ 0 & \eta(1-F) & -(\eta A + \mu) & 0 \\ 0 & \lambda[1 + \kappa(1-F)] & -\lambda\kappa A & -\delta \end{pmatrix}. \quad (5)$$

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The entries have direct interpretations: soil is depleted by productivity ($-\alpha S$) and by degradation pressure ($-\chi S$); better soil raises productivity growth ($A\beta(1-A)$); productivity is self-limited by capacity ($-AB$), reduced by degradation ($-\gamma A$) and by food security through reduced expansion ($-A\xi(1-A)$); productivity raises food security ($\eta(1-F)$) which saturates and decays ($-(\eta A+\mu)$); activity generates pressure ($\lambda[1+\kappa(1-F)]$), higher food security relieves it ($-\lambda\kappa A$) and it dissipates ($-\delta$). The zero entries show that soil does not directly affect food security or degradation pressure, and that food security and degradation pressure do not directly affect soil beyond the stated terms.

At E^*, $\Gamma=0$, and the equilibrium identity $\alpha A+\chi D=(1-S)\left(r_s S+\rho\right)/S$ reduces the first diagonal entry to $-\left(r_s S^*+\rho/S^*\right)$. Put

$$P=r_s S^*+\rho/S^*,\ Q=A\beta(1-A),\ W=A\xi(1-A),\ N=\eta A+\mu,$$

$u=\eta(1-F)$, $v=\lambda[1+\kappa(1-F)]$ and $z=\lambda\kappa A$, all evaluated at E^*. Then

$$J(E^*)=\begin{pmatrix} -P & -\alpha S & 0 & -\chi S \\ Q & -AB & -W & -\gamma A \\ 0 & u & -N & 0 \\ 0 & v & -z & -\delta \end{pmatrix}. \tag{6}$$

The characteristic polynomial is $\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4$, with $a_1=-trJ$, a_2 the sum of the six 2×2 principal minors, a_3 minus the sum of the four 3×3 principal minors and $a_4=detJ$:

$$\begin{aligned} a_1 &= P+AB+N+\delta, \\ a_2 &= AB(N+\delta)+N\delta+P(AB+N+\delta)+\alpha SQ+Wu+\gamma Av, \\ a_3 &= ABN\delta+N\alpha SQ+N\gamma Av+P\left(ABN+AB\delta+N\delta+Wu+\gamma Av\right) \\ &\quad +\alpha SQ\delta+\chi SQv+W\delta u-\gamma A uz, \\ a_4 &= ABNP\delta+P\gamma A(Nv-uz)+\alpha SQN\delta+\chi SQ(Nv-uz)+PW\delta u. \end{aligned} \tag{7}$$

These closed forms were verified against the numerically computed characteristic polynomial and against symbolic expansion (Section 5.1).

4.3 Local stability

Lemma 1. $\Delta:=Nv-uz=\lambda[\eta A+\mu(1+\kappa(1-F))]>0$.

Proof. $Nv-uz=(\eta A+\mu)\lambda(1+\kappa(1-F))-\eta(1-F)\lambda\kappa A$. Expanding, the terms $\kappa\eta A(1-F)$ cancel leaving $\lambda[\eta A+\mu(1+\kappa(1-F))]$.

Theorem 3 (Stability criterion). For every admissible parameter set with $R_A>1$, $a_1,a_2,a_3,a_4>0$. Consequently E^* is locally asymptotically stable if and only if

$$\Delta_3:=a_3\left(a_1a_2-a_3\right)-a_1^2a_4>0, \tag{8}$$

and E^* can lose stability only through a pair of purely imaginary eigenvalues (a Hopf bifurcation), never through a zero eigenvalue.

Proof. $a_1,a_2>0$ is immediate. In a_3 the only negative term is $-\gamma A uz$, and a_3 contains $N\gamma Av$; since $N\gamma Av-\gamma A uz=\gamma A\Delta>0$ by Lemma 1, $a_3>0$. In a_4 the two potentially negative terms are absorbed

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as $P\gamma A\Delta$ and $\chi SQ\Delta$, so $a_4 > 0$. For a quartic with positive coefficients, the Routh–Hurwitz conditions [8] reduce to $a_1 a_2 > a_3$ and $a_3\left(a_1 a_2 - a_3\right) > a_1^2 a_4$, and the second implies the first since $a_1^2 a_4 > 0$. Since $a_4 = detJ > 0$, zero is never an eigenvalue. ▫

Corollary 2 (Local classification). If $\Delta_3 > 0$ all four eigenvalues have negative real part, E^* is hyperbolic and by the Hartman–Grobman theorem [21,19] the flow near E^* is conjugate to the linear flow: a stable node or focus. If $\Delta_3 < 0$, E^* is unstable.

The only destabilising term in the coefficients is $-\gamma A\, uz$: the chain productivity → food security → degradation → productivity, that is, the feedback loop of the Introduction.

4.4 Eigenvalues and Eigenvectors at *E*

$J\left(E_0\right)$ has eigenvalues $-\left(r_s + \rho\right)$, $\Lambda - \theta$, $-\mu$ and $-\delta$. With $\sigma = r_s + \rho$ and $\varphi = \Lambda - \theta$, solving $\left(J\left(E_0\right) - \nu I\right)v = 0$ row by row gives the eigenvectors $(1,0,0,0)^{\top}$ for $-\sigma$; $(0,0,1,0)^{\top}$ for $-\mu$ (if $\mu \neq \sigma$); $(\chi/(\delta - \sigma), 0, 0, 1)^{\top}$ for $-\delta$ (if $\delta \neq \sigma$); and

$$\left(-\frac{\alpha+\chi v_D}{\varphi+\sigma},\ 1,\ \frac{\eta}{\varphi+\mu},\ v_D\right)^{\top}, \qquad v_D = \frac{\lambda(1+\kappa)}{\varphi+\delta}, \tag{9}$$

for φ. At the baseline the eigenvalues are -0.35, $+1.00$, -0.30 and -0.40, with eigenvectors $(1,0,0,0)$, $(-0.3598, 1, 0.6154, 0.7143)$, $(0,0,1,0)$ and $(1,0,0,0.125)$ respectively. The unstable direction activates productivity first, then raises food security and degradation pressure together, while soil quality falls.

4.5 Global stability

Theorem 4 (Global stability of E_0**).** E_0 is locally asymptotically stable if $R_A < 1$ and a saddle if $R_A > 1$. If $R_A \leq 1$, E_0 is globally asymptotically stable in Ω.

Proof. The local statements follow from the eigenvalues. For global stability, $B \leq \Lambda$ on Ω, so $\dot{A} \leq A[\Lambda(1 - A) - \theta] = A[(\Lambda - \theta) - \Lambda A \leq -\Lambda A^2$ when $\Lambda \leq \theta$,

giving $A(t) \leq A_0/\left(1 + \Lambda A_0 t\right) \to 0$. Then $\dot{F} \leq \eta A(t) - \mu F$ gives $F \to 0$ and $\dot{D} \leq \lambda(1 + \kappa)A(t) - \delta D$ gives $D \to 0$. For S, for any $\varepsilon > 0$ eventually $\alpha A + \chi D < \varepsilon$, so $\dot{S} \geq (1 - S)\left(r_s S + \rho\right) - \varepsilon S$, whose positive equilibrium tends to 1 as $\varepsilon \to 0$; with $S \leq 1$ this gives $S \to 1$. ▫

For E^* the standard Lyapunov candidate $V = \sum_{i=1}^{4} c_i\left[x_i - x_i^* - x_i^* ln\left(x_i/x_i^*\right)\right]$ [12,13] does not give $\dot{V} \leq 0$. The pairs (S, A), (A, F) and (A, D) have off-diagonal entries of opposite sign, so weights can cancel their cross terms, but the couplings from D to S and from F to D are one-way (the reciprocal entries are zero), so their cross terms cannot be cancelled and $\dot{V}$ is not sign-definite. More fundamentally, Section 4.6 exhibits a stable limit cycle, which shows that no global Lyapunov function exists for all parameters. The strongest rigorous results are therefore global stability of E_0 when $R_A \leq 1$, local stability of E^* if and only if $\Delta_3 > 0$, and a local quadratic Lyapunov function: at the baseline the solution P of $J^{\top}P + PJ = -I$ is symmetric

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positive definite (eigenvalues 0.783, 1.172, 2.232, 3.821), so $V = \left(x - x^*\right)^{\top} P\left(x - x^*\right)$ satisfies $\dot{V} = -\|x - x^*\|^2 + O\left(\|x - x^*\|^3\right)$ near E^*. Global attractivity of E^* at the baseline is supported numerically only: all 500 random initial states in Ω converged to E^*.

4.6 Bifurcation analysis

A zero eigenvalue at E^* is impossible, since $detJ\left(E^*\right) = a_4 > 0$. Hence no saddle-node, pitchfork or transcritical bifurcation of E^* with another interior equilibrium can occur, and multiple equilibria are excluded. The only bifurcation of E_0 occurs at $\Lambda = \theta$, and the only bifurcation of E^* is Hopf.

Theorem 5 (Forward transcritical bifurcation). Let $\sigma = r_s + \rho$, $g = \alpha + \chi\lambda(1 + \kappa)/\delta$ and

$$c = \Lambda + \frac{\beta g}{\sigma} + \frac{\xi\eta}{\mu} + \frac{\gamma\lambda(1+\kappa)}{\delta} > 0. \tag{10}$$

On the centre manifold of E_0 at $\theta = \Lambda$, $\dot{A} = (\Lambda - \theta)A - cA^2 + O\left(A^3\right)$. Hence for $\theta < \Lambda$ a stable positive equilibrium with $A^* \approx (\Lambda - \theta)/c$ branches off from E_0, which becomes a saddle.

Proof. At $\theta = \Lambda$, $J\left(E_0\right)$ has eigenvalues $0, -\sigma, -\mu, -\delta$, so a one-dimensional centre manifold exists, parametrised by A. On it, to first order, $F = (\eta/\mu)A$, $D = (\lambda(1 + \kappa)/\delta)A$ and $S - 1 = -(g/\sigma)A$. With $B = \Lambda + \beta(S - 1) - \xi F$, substituting into $\dot{A} = A[B(1 - A) - \gamma D - \theta]$ gives $\dot{A} = A\left[(\Lambda - \theta) - cA + O\left(A^2\right)\right]$. ▫

At the baseline $c = 5.140$ and the threshold is $\theta_c = 1.3$; the exact equilibrium agrees with the prediction (at $\theta = 1.29$, $A^* = 0.00195$ against 0.00195; at $\theta = 1.25$, 0.00989 against 0.00973). Figure 2 shows the bifurcation.

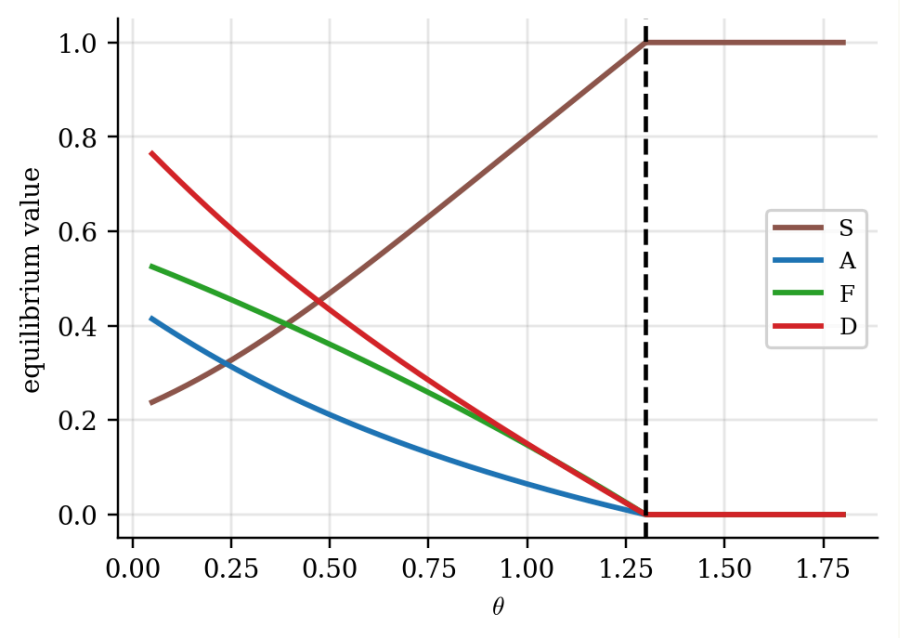


Figure 2. Transcritical bifurcation at $R_A = 1$ ($\theta = 1.3$). The productive branch exists and is stable for $\theta < 1.3$; above the threshold only E_0 remains and it is globally stable.

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Theorem 6 (Hopf criterion). Let a parameter ϵ vary and suppose $\Delta_3(\epsilon_H) = 0$ with $d\Delta_3/d\epsilon(\epsilon H) \neq 0$ there. Then $J(E^*)$ has exactly one pair of purely imaginary eigenvalues $\pm i\omega$ with $\omega^2 = a_3/a_1$, its other two eigenvalues have negative real part, and a Hopf bifurcation occurs [9,15].

Proof. Since $a_i > 0$, suppose the polynomial factors as $(\lambda^2 + \omega^2)(\lambda^2 + p\lambda + q)$. Matching coefficients gives $p = a_1$, $a_2 = q + \omega^2$, $a_3 = p\omega^2$ and $a_4 = q\omega^2$, so $\omega^2 = a_3/a_1$ and $q = a_1 a_4/a_3 > 0$. Equating $q = a_2 - a_3/a_1$ with $q = a_1 a_4/a_3$ gives $a_1 a_2 a_3 - a_3^2 = a_1^2 a_4$, i.e. $\Delta_3 = 0$; conversely $\Delta_3 = 0$ with $a_i > 0$ yields the factorisation. Because $p, q > 0$, the second factor has roots with negative real part. Nonzero transversality of Δ_3 is equivalent to nonzero crossing speed [6,15]. ▫

Bifurcation in α and χ. At the baseline, varying α over $[0, 5]$ leaves the largest real part of the spectrum below -0.287, so there is no bifurcation in α (Figure 3a). In a stress-test set P_H ($r_s = 1.7$, $\rho = 0.01$, $\alpha = 0.15$, $\chi = 0.9$, $r_a = 0.3$, $\beta = 2.2$, $\gamma = 0.8$, $\xi = 0.3$, $\theta = 0.5$, $\eta = 0.5$, $\mu = 1.3$, $\lambda = 2.0$, $\kappa = 3.6$, $\delta = 0.1$; $R_A = 5.6$), two Hopf bifurcations were verified. With $\alpha = 0.15$, $\chi_H = 0.8071$: $E^* = (0.4167, 0.01357, 0.00519, 1.2433)$, eigenvalues $\pm 0.4394i$, $\Delta_3 = 0$ to machine precision, $\omega^2 = a_3/a_1 = 0.1930$ equal to the squared imaginary part, and transversality $dRe\lambda/d\chi = 0.0628 > 0$. With $\chi = 0.9$, $\alpha_H = 1.3850$: $E^* = (0.3852, 0.0127, 0.0048, 1.1596)$, eigenvalues $\pm 0.4370i$, $\omega^2 = 0.1910$, and $dRe\lambda/d\alpha = -0.0044 < 0$, so E^* is unstable for $\alpha < \alpha_H$ and stable above (Figure 3b). Simulation shows a stable limit cycle for $\chi > \chi_H$ whose amplitude grows continuously from zero, so the bifurcation is numerically supercritical (the first Lyapunov coefficient was not computed analytically); the period is about 14.3 years, and at $\chi = 0.9$ productivity oscillates between 0.0020 and 0.0384 (Figure 4). Table 2 gives the critical eigenvectors: the oscillation is carried by the slow degradation-pressure and soil variables. Oscillation requires slow dissipation of degradation pressure (δ small) and strong dependence of productivity on soil (β large); it is a boom–bust regime at very low productivity ($A^* \approx 0.014$), and arises only in a stress-test set whose empirical relevance is unestablished.

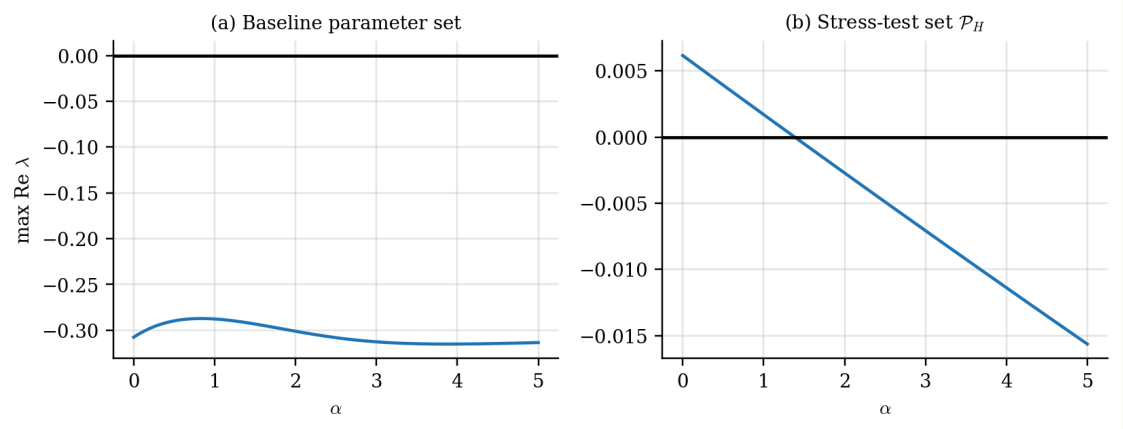


Figure 3. Largest real part of the spectrum of $J(E^*)$ as α varies: (a) baseline set, no bifurcation; (b) stress-test set P_H ($\chi = 0.9$), Hopf bifurcation at $\alpha_H = 1.385$.

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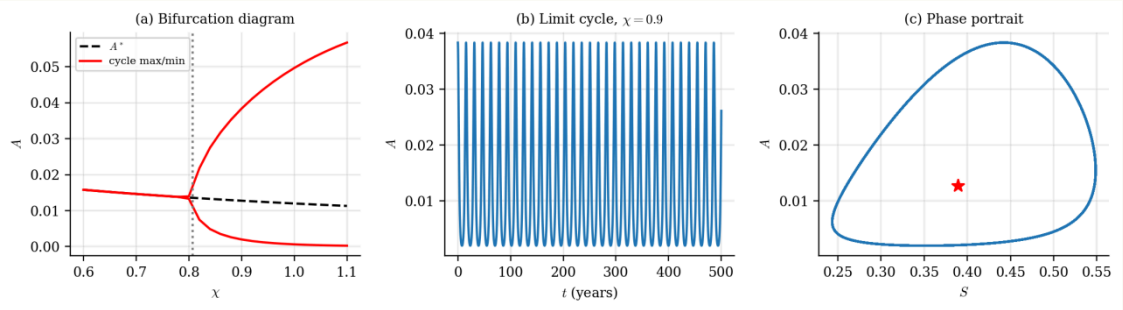


Figure 4. Supercritical Hopf bifurcation in χ for the stress-test set P_H: (a) bifurcation diagram (dashed: equilibrium; solid: extrema of the limit cycle); (b) productivity limit cycle at $\chi = 0.9$; (c) phase portrait in (S, A).

Table 2. Eigenvalues and eigenvectors at the Hopf point $\chi = \chi_H = 0.8071$ of the stress-test set P_H; $E^* = (0.4167, 0.01357, 0.00519, 1.2433)$.

λ	v_S	v_A	v_F	v_D
0.0000 + 0.4394i	-0.3402 + 0.2000i	0.0110 + 0.0481i	0.0093 + 0.0152i	1.0000
-0.8544	1.0000	-0.0307	-0.0337	0.3683
-1.3054	0.0353	0.0028	1.0000	0.0596

5. NUMERICAL RESULTS

5.1 Parameters, scenarios and methods

All parameter values are hypothetical (Table 1); the literature supports the direction of each effect, not its magnitude. Scenarios modify the baseline as in Table 3; none changes R_A, so all remain in the productive regime.

Table 3. Management scenarios.

Scenario	Parameter change
A: business as usual	none
B: soil restoration	$\rho: 0.05 \rightarrow 0.20$
C: sustainable management	$\alpha: 0.20 \rightarrow 0.10$, $\chi: 0.40 \rightarrow 0.20$, $\lambda: 0.50 \rightarrow 0.30$
D: combined	B and C together
Low degradation pressure	$\lambda: 0.50 \rightarrow 0.25$
High degradation pressure	$\lambda: 0.50 \rightarrow 1.00$
Unsustainable practice	$\lambda = 1.0, \alpha = 0.30, \chi = 0.60, \rho = 0.01$

Initial states for baseline time series are IC1 $= (0.8, 0.5, 0.6, 0.1)$, IC2 $= (0.3, 0.2, 0.2, 1.0)$ and IC3 $= (0.95, 0.9, 0.9, 0)$; scenario comparisons use IC1. The system was integrated in Python (SciPy, LSODA) with relative tolerance 10^{-10} and absolute tolerance 10^{-12}. Equilibria were obtained by root-finding on Φ, eigenvalues and eigenvectors from the analytic Jacobian. Verification: at every simulated parameter set the equilibrium residual was at most 7.8×10^{-16}; the analytic Jacobian matched

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a finite-difference Jacobian to within 3.8×10^{-10}; the closed-form coefficients $a_1, ..., a_4$ matched the numerical polynomial to within 2.4×10^{-15}; and $max_i \| Jv_i - \lambda_i v_i \| \leq 7.2 \times 10^{-16}$. In a stress test with random parameters (each baseline value multiplied by a random factor in $[0.05, 6]$), among 19,690 sets with $R_A > 1$ the unique root of Φ was found in every case and $a_1, ..., a_4 > 0$ held in every case; the interior equilibrium was unstable in 34 sets (0.17%), always through a complex pair. Code is available on request.

5.2 Baseline dynamics and eigenstructure

At the baseline, $R_A = 4.333$ and $E^* = (0.3530, 0.2911, 0.4370, 0.5688)$. The eigenvalues are $-0.2980 \pm 0.3463i$, -0.3731 and -0.4514, all with negative real part and $\Delta_3 = 0.1910 > 0$, so E^* is locally asymptotically stable. The oscillatory mode has period $2\pi/0.3463 \approx 18.1$ years and decay time about 3.4 years. All three initial states converge to E^* (Figures 5 and 6), and 14 random initial states in Ω all approach the single attractor with the spiral structure predicted by the complex pair (Figure 7).

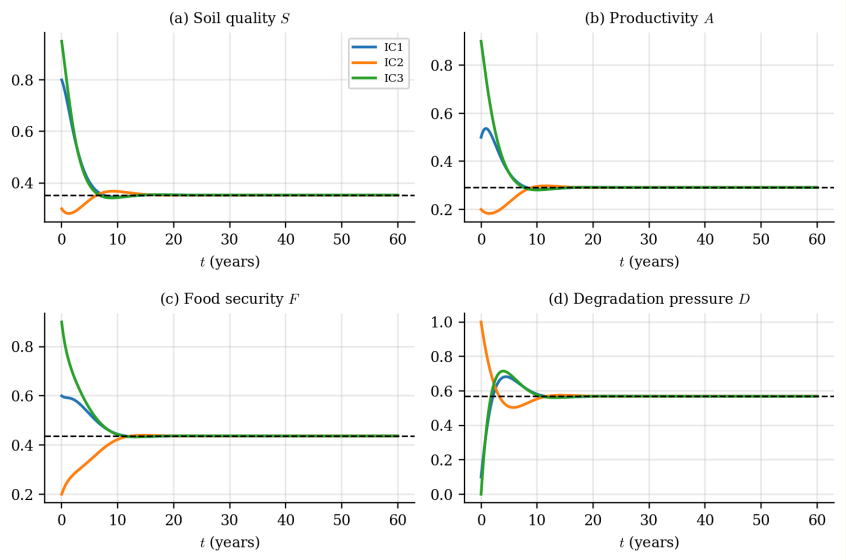


Figure 5. *Baseline time series from three initial states: (a) soil quality, (b) productivity, (c) food security, (d) degradation pressure. Dashed lines mark E^*.*

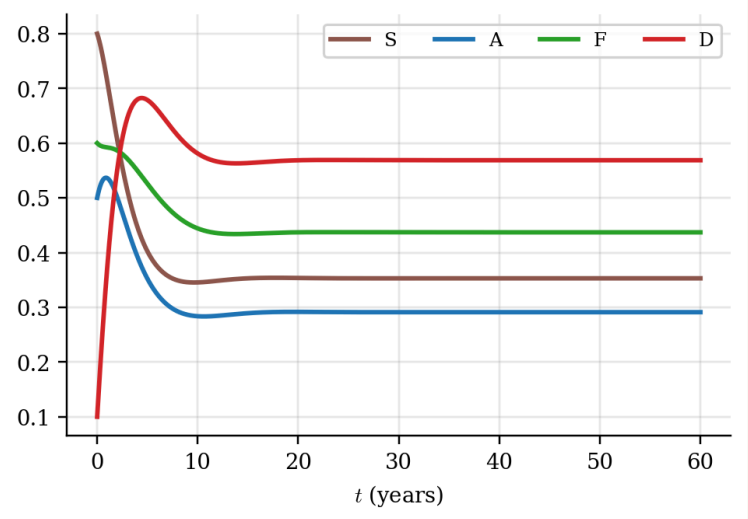


Figure 6. Combined baseline trajectories of S, A, F and D from IC1.

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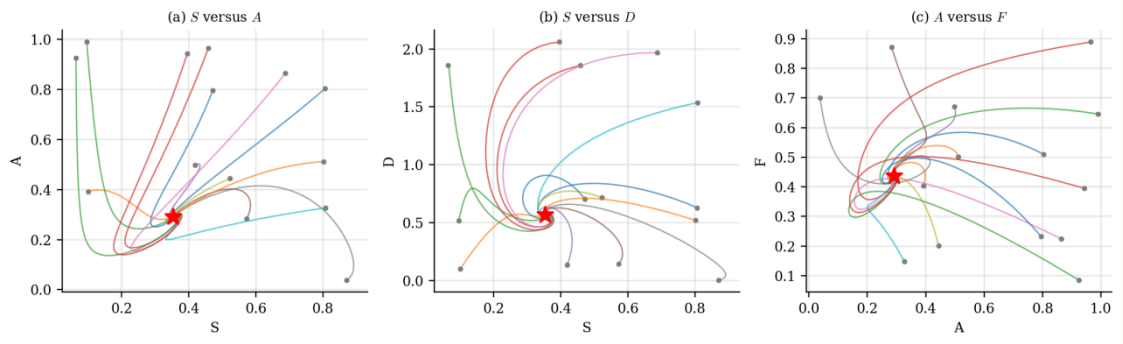


Figure 7. Phase planes for 14 random initial states: (a) S versus A, (b) S versus D, (c) A versus F. The red star is E^*.

Tables 5–8 give the eigenvalues and eigenvectors in each scenario (components ordered S, A, F, D). In every scenario there are two real modes and one oscillatory mode. Under business as usual and sustainable management the slowest real mode has its largest component in S with D strongly involved (component 0.86 in scenario A): a slow soil–degradation recovery in which a soil perturbation relaxes together with the degradation pressure that damaged it. The other real mode is led by F and is nearly decoupled from the rest, relaxing at a rate of the order of $\eta A^* + \mu = 0.533$. The complex pair involves all four variables with a phase lag between S and F, so perturbations spiral back to E^*. Under restoration the soil-led mode becomes the fastest real mode (-0.6417). The dominant decay rate is 0.298, 0.348, 0.351 and 0.462 in scenarios A to D, so the combined policy also returns fastest from a perturbation.

Table 5. Eigenvalues and eigenvectors at E^*, scenario A (business as usual).

$E^* = (0.3530, 0.2911, 0.4370, 0.5688)$***; characteristic polynomial***

$\lambda^4 + 1.4205\lambda^3 + 0.8685\lambda^2 + 0.2724\lambda + 0.03515$***. Eigenvectors are scaled so that the largest component equals 1; each complex eigenvalue is listed once (its conjugate has the conjugate eigenvector).***

λ	v_S	v_A	v_F	v_D
-0.2980 + 0.3463i	-0.0261 + 0.4589i	0.2323 + 0.4559i	0.5464 + 0.0685i	1.0000
-0.3731	1.0000	0.0622	0.1753	0.8578
-0.4514	0.1178	0.1810	1.0000	0.0795

Table 6. *Eigenvalues and eigenvectors at E^*, scenario B (soil restoration, $\rho = 0.20$).*

$E^* = (0.5284, 0.3315, 0.4692, 0.6343)$*; characteristic polynomial*

$\lambda^4 + 1.8083\lambda^3 + 1.3639\lambda^2 + 0.5301\lambda + 0.08679$*. Eigenvectors are scaled so that the largest component equals 1; each complex eigenvalue is listed once (its conjugate has the conjugate eigenvector).*

λ	v_S	v_A	v_F	v_D
-0.3484 + 0.4081i	-0.3330 + 0.4109i	0.1815 + 0.5530i	0.5270 + 0.0910i	1.0000
-0.4696	-0.0600	0.2250	1.0000	-0.0934
-0.6417	1.0000	-0.0765	0.4247	0.5336

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Table 7. *Eigenvalues and eigenvectors at E^*, scenario C (sustainable management).* $E^* = (0.6291, 0.4413, 0.5406, 0.4831)$*; characteristic polynomial* $\lambda^4 + 1.7491\lambda^3 + 1.2330\lambda^2 + 0.4110\lambda + 0.05281$*. Eigenvectors are scaled so that the largest component equals 1; each complex eigenvalue is listed once (its conjugate has the conjugate eigenvector).*

λ	v_S	v_A	v_F	v_D
-0.3509	1.0000	0.1071	0.1302	0.6040
-0.4297 + 0.3077i	0.0515 + 0.4122i	0.1483 + 0.8061i	0.7147 + 0.3417i	1.0000
-0.5388	0.0597	0.3110	1.0000	-0.0270

Table 8. *Eigenvalues and eigenvectors at E^*, scenario D (combined intervention).* $E^* = (0.7411, 0.4656, 0.5539, 0.5050)$*; characteristic polynomial* $\lambda^4 + 2.0461\lambda^3 + 1.6799\lambda^2 + 0.6593\lambda + 0.10328$*. Eigenvectors are scaled so that the largest component equals 1; each complex eigenvalue is listed once (its conjugate has the conjugate eigenvector).*

λ	v_S	v_A	v_F	v_D
-0.4624 + 0.3401i	-0.2337 + 0.4347i	0.0891 + 0.8978i	0.7238 + 0.3535i	1.0000
-0.5311	-0.1819	0.3962	1.0000	-0.2458
-0.5901	0.4896	0.2308	1.0000	0.2080

5.3 Scenario analysis

Table 4. Equilibria and local stability for all simulated scenarios ($R_A = 4.333$ ***in every case).***

Scenario	S^*	A^*	F^*	D^*	$maxRe\lambda$	Δ_3
A: business as usual	0.3530	0.2911	0.4370	0.5688	-0.2980	0.1910
B: soil restoration	0.5284	0.3315	0.4692	0.6343	-0.3484	0.7425
C: sustainable management	0.6291	0.4413	0.5406	0.4831	-0.3509	0.5558
D: combined	0.7411	0.4656	0.5539	0.5050	-0.4624	1.3991
Low degradation pressure	0.4276	0.4203	0.5285	0.3865	-0.3662	0.3681
High degradation pressure	0.2949	0.1761	0.3195	0.7397	-0.2396	0.1053
Unsustainable practice	0.0765	0.1373	0.2680	0.5945	-0.2099	0.0399

Halving degradation-pressure generation ($\lambda = 0.25$) raises soil quality by 21%, productivity by 44% and food security by 21% and lowers degradation pressure by 32%; doubling it lowers soil quality by 16%, productivity by 40% and food security by 27% and raises pressure by 30%, with a smaller dominant decay rate (0.240 against 0.298) and a smaller Δ_3 (0.105 against 0.191), so recovery is lower and slower although still stable (Figure 8).

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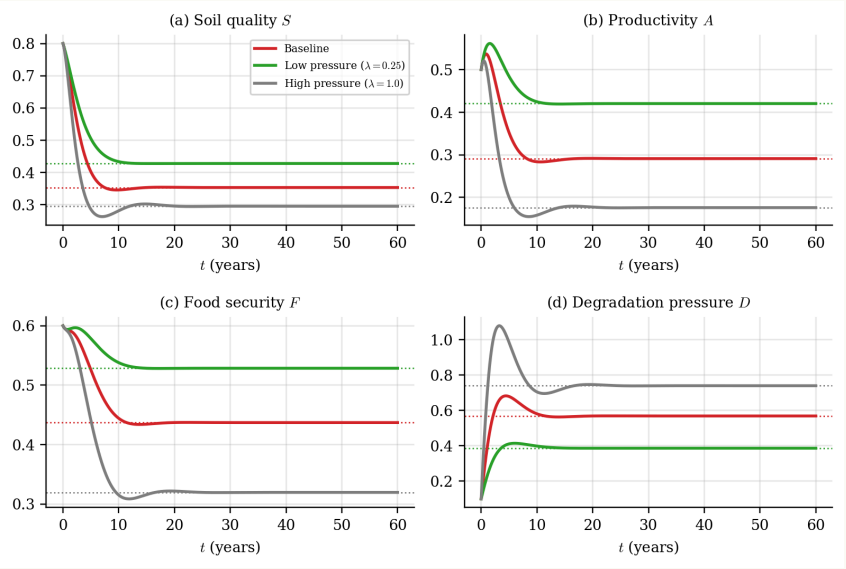


Figure 8. Effect of degradation pressure: baseline, low pressure ($\lambda = 0.25$) and high pressure ($\lambda = 1.0$).

Restoration alone (B, $\rho = 0.20$) raises soil quality by 50%, productivity by 14% and food security by 7%, but raises degradation pressure by 12%: because $D^* \propto A^*$, a more productive system generates more activity-driven pressure, so restoration alone does not reduce pressure. Sustainable management (C) raises soil quality by 78%, productivity by 52% and food security by 24% and lowers pressure by 15%. The combined scenario (D) gives $+110\%$ soil quality, $+60\%$ productivity, $+27\%$ food security and -11% pressure, with the fastest recovery (food security within 2% of equilibrium after 7.8 years against 9.8 years at baseline). The gains are sub-additive: food security rises 27% against 31% for the separate gains added, and soil quality 110% against 128% (Figure 9). Relative to the unsustainable system ($E^* = (0.0765, 0.1373, 0.2680, 0.5945)$), the sustainable one has 9.7 times the soil quality, 3.4 times the productivity and 2.1 times the food security (Figure 10). These differences are all within the productive regime; they reflect different equilibria, not different regimes. Figure 11 shows that F^* and S^* increase monotonically as degradation intensity falls and as restoration rises over the whole grid, with the degradation multiplier having the larger effect.

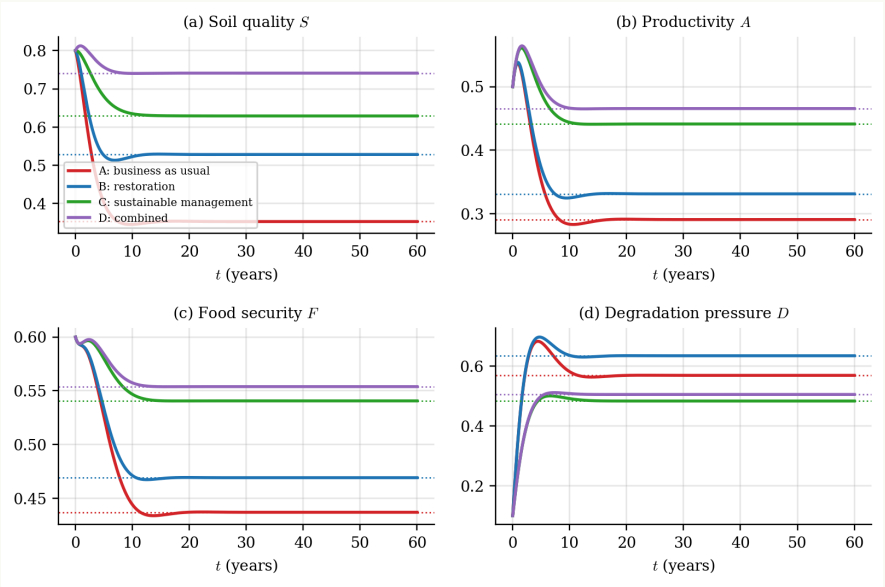


Figure 9. Scenarios A–D: (a) soil quality, (b) productivity, (c) food security, (d) degradation pressure.

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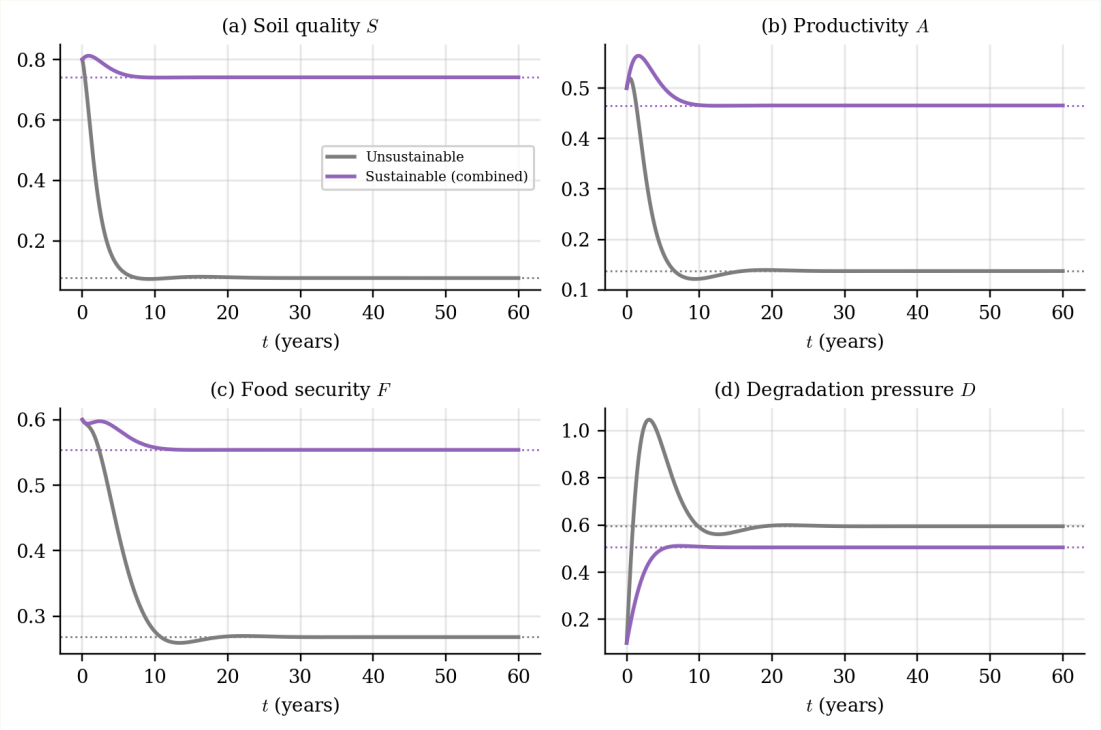


Figure 10. Sustainable (combined) versus unsustainable practice.

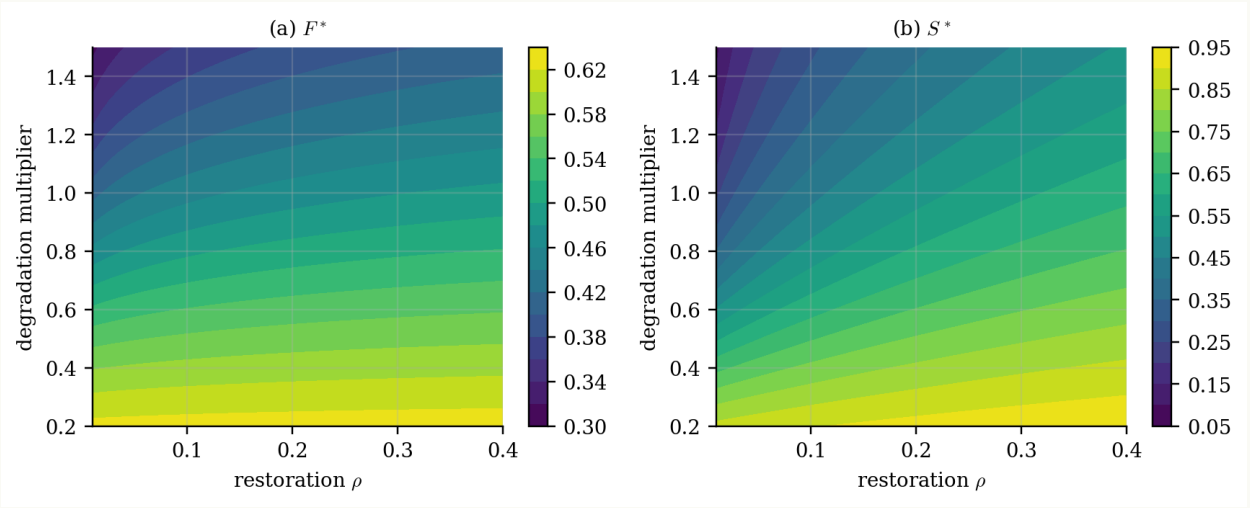


Figure 11. Equilibrium (a) food security F^* and (b) soil quality S^* as functions of restoration ρ and a multiplier on (α, χ, λ).

5.4 Sensitivity analysis

Local normalised forward indices were computed by central differences, and global sensitivity by Latin hypercube sampling (2000 samples, each parameter varied $\pm$ 50%) with partial rank correlation coefficients (Table 9; Figure 12). Food security F^* is most sensitive to η and μ ($\pm$ 0.59), then λ and δ ($\mp$ 0.36), r_a ($+$ 0.30) and θ ($-$ 0.25). Productivity A^* is most sensitive to λ and δ ($\mp$ 0.63), r_a ($+$ 0.53), θ ($-$ 0.45) and γ ($-$ 0.43). Soil quality S^* is most sensitive to χ ($-$ 0.71), r_s ($+$ 0.61), r_a ($-$ 0.54), θ ($+$ 0.46) and γ ($+$ 0.43). Degradation pressure D^* is most sensitive to λ and δ ($\pm$ 0.47), r_a ($+$ 0.45), θ ($-$ 0.38) and γ ($-$ 0.36). The equal and opposite entries for (η, μ) and (λ, δ) are exact consequences of Corollary 1. Local and global rankings agree on the leading parameters; PRCC magnitudes are larger because they measure rank correlation over $\pm$ 50% ranges rather than a derivative at one point. The negative effect of r_a on S^* is structural: higher productivity growth raises activity-driven degradation.

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Table 9. Normalised forward sensitivity indices (SI) at the baseline equilibrium and partial rank correlation coefficients (PRCC; Latin hypercube sampling, 2000 admissible samples, each parameter varied ±50%).

Parameter	SI S^*	SI A^*	SI F^*	SI D^*	PRCC S^*	PRCC A^*	PRCC F^*	PRCC D^*
r_s	+0.606	+0.177	+0.099	+0.149	+0.850	+0.418	+0.296	+0.390
ρ	+0.286	+0.083	+0.047	+0.070	+0.608	+0.208	+0.132	+0.180
α	-0.182	-0.053	-0.030	-0.045	-0.481	-0.140	-0.082	-0.128
χ	-0.710	-0.207	-0.117	-0.175	-0.886	-0.494	-0.331	-0.455
r_a	-0.537	+0.532	+0.300	+0.448	-0.824	+0.841	+0.717	+0.824
β	-0.228	+0.225	+0.127	+0.190	-0.551	+0.567	+0.409	+0.529
γ	+0.431	-0.427	-0.240	-0.360	+0.735	-0.767	-0.618	-0.743
ξ	-0.121	+0.120	+0.067	+0.101	-0.318	+0.338	+0.239	+0.330
θ	+0.455	-0.450	-0.254	-0.379	+0.770	-0.793	-0.636	-0.776
η	+0.097	+0.047	+0.590	-0.117	+0.219	+0.107	+0.896	-0.333
μ	-0.097	-0.047	-0.590	+0.117	-0.263	-0.103	-0.896	+0.354
λ	-0.279	-0.634	-0.357	+0.466	-0.587	-0.875	-0.759	+0.841
κ	-0.100	-0.228	-0.129	+0.168	-0.248	-0.515	-0.375	+0.450
δ	+0.279	+0.634	+0.357	-0.466	+0.599	+0.876	+0.761	-0.830

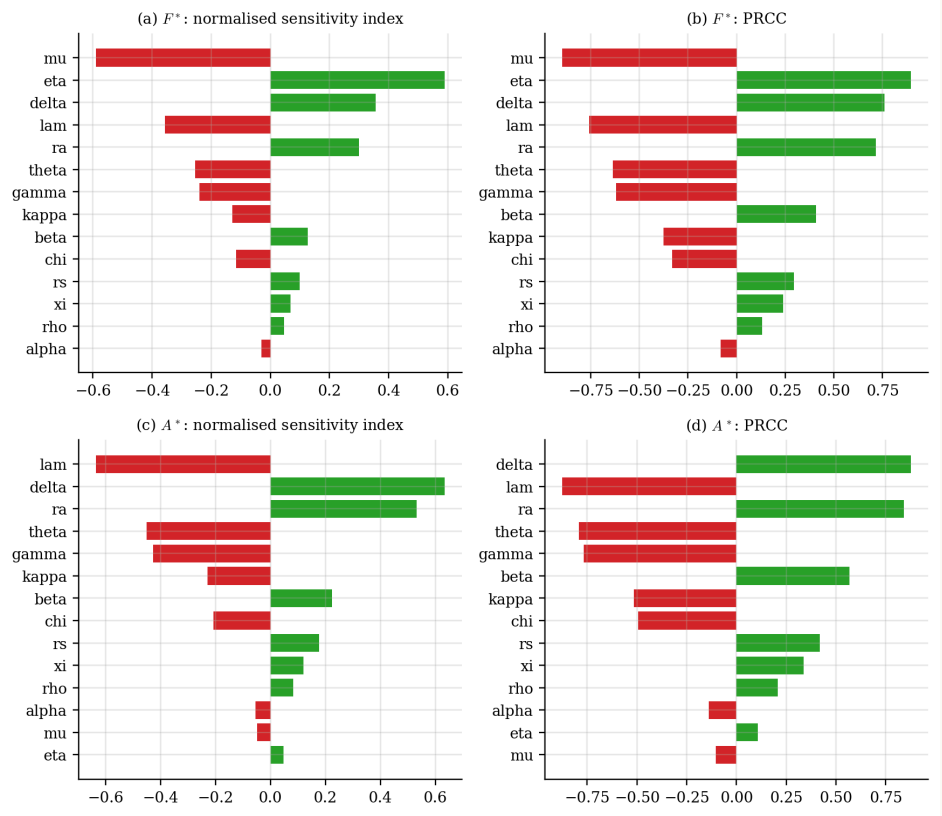


Figure 12. Sensitivity of (a, b) food security F^* and (c, d) productivity A^*: normalised sensitivity indices (left) and PRCC (right).

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6. DISCUSSION

- **Threshold behaviour:** When $R_A > 1$ the system settles to a unique positive equilibrium; when $R_A \leq 1$ production dies out. The reason is that Φ is strictly decreasing, so it has a root exactly when $\Phi(0) = \Lambda - \theta > 0$, and a comparison argument gives global extinction otherwise. A system whose productive growth cannot outpace its losses cannot sustain production regardless of soil quality, and no amount of restoration changes this; only R_A does. Threshold numbers of this kind are standard in epidemic and ecological modelling [25]; their use for productivity here is a modelling choice.
- **Feedback and instability:** At the baseline the feedback loop is damped, giving a spiral approach to E^*; in stress-test sets it can oscillate. The loop through food security and degradation enters the stability criterion only through $-\gamma A\, uz$, and instability requires $\Delta_3 < 0$. Boom–bust cycles of productivity are possible when degradation pressure lingers; the model gives no evidence that they occur in practice.
- **Interventions:** Reducing degradation intensity outperformed restoration alone for productivity, food security and pressure, and combining the two gave the highest levels but sub-additive gains. Restoration raises soil quality, which raises productivity and hence degradation pressure through $D^* \propto A^*$, offsetting part of the gain; reducing intensity attacks the generation of pressure itself. The direction is consistent with dynamical models in which input-led intensification can degrade soil quality and in which combinations and sequencing of interventions matter [22], and with field evidence that conservation practices improve soil health without lowering yields [24]. This is qualitative agreement only; the model has not been validated against data. Relative to asset-compartment models of agricultural dynamics [1,2], the present model adds soil quality, a food-security index and an explicit pressure feedback, and offers closed-form threshold and stability results.
- **Limitations:** (i) Soil is a single index; physical, chemical and biological processes and nutrient-specific dynamics are not resolved. (ii) All parameters are hypothetical and none was calibrated, so quantitative results illustrate mechanism and are not predictions. (iii) The variables are dimensionless and cannot be read as tonnes, hectares or calories without a mapping to data. (iv) Climate variability, price changes and policy shocks are excluded. (v) The model is a single aggregate system, with no spatial or regional heterogeneity. (vi) The link from food insecurity to production pressure (ξ and κ) is a hypothesis. (vii) F is a single composite index; access, utilisation and stability are not separately modelled. (viii) No data were used. (ix) Global stability of E^* was not proved, the first Lyapunov coefficient of the Hopf bifurcation was not computed analytically, and the Hopf results come from a stress-test parameter set. (x) The model is monostable, so it cannot represent tipping or bistability.

7. CONCLUSION, RECOMMENDATIONS AND FURTHER RESEARCH

7.1 Conclusion

A four-variable nonlinear model for normalised soil quality, productivity, food security and degradation pressure was constructed and analysed. The system has the productivity-free equilibrium $E_0 = (1, 0, 0, 0)$ always, and a unique positive equilibrium if and only if $R_A = \left(r_a + \beta + \xi\right)/\theta > 1$; if $R_A \leq 1$, E_0 attracts every solution. At E^* the Jacobian has a negative diagonal and a characteristic polynomial with four positive coefficients for all admissible parameters, so E^* is locally asymptotically stable if and only if

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$\Delta_3 > 0$ and can lose stability only by Hopf bifurcation; the transcritical bifurcation at $R_A = 1$ is forward. Eigenvalues and eigenvectors were obtained at E_0 analytically and at E^* for four scenarios; at the baseline the eigenvalues are $-0.298 \pm 0.346i$, -0.373 and -0.451. Food security is controlled mainly by the ratios η/μ and λ/δ and by r_a and θ; soil quality by χ and r_s. In the hypothetical baseline, reducing degradation intensity is more effective than restoration alone, and the combination gives the highest levels (soil +110%, productivity +60%, food security +27%) with sub-additive gains and the fastest recovery. These are theorems and simulations of the model as formulated; they are not statements about any real system until the model is calibrated and tested.

7.2 Recommendations

- ***For modelling:*** Calibrate the identifiable combinations (η/μ, λ/δ, $\chi\lambda/\delta$, $\gamma\lambda/\delta$) before interpreting magnitudes; extend the soil–productivity response or the regeneration term nonlinearly if bistability is of interest; compute the first Lyapunov coefficient and derive Lyapunov-based sufficient conditions for global stability in restricted regions; and develop an optimal-control formulation with controls for restoration effort and sustainable management once calibrated parameters are available.
- ***For agricultural systems (conditional on the model's assumptions; not policy advice):*** Within the model, measures that lower the degradation pressure generated per unit of production are more effective than restoration alone, and the two are complementary; productivity cannot be sustained when growth potential does not exceed losses ($R_A \leq 1$), so the balance between growth and loss, not only soil condition, is relevant; and because the benefit of combined measures is sub-additive, expected gains from stacking interventions should not be extrapolated by addition. Evaluating any of these in a real system requires local calibration.

7.3 Further research

Natural extensions include calibration to regional data, climate and price forcing, and separate treatment of the access, utilisation and stability dimensions of food security. Memory effects in soil processes and spatial transport can be represented with time-fractional and advection–diffusion formulations, for which wavelet-based Galerkin discretisations are available [10,11]. A stochastic version of the model, with long-run behaviour described through invariant measures, can draw on ergodic-theoretic tools for measure-preserving transformations [19,20].

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