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An Extended Framework For Overcoming The Radiation Pressure Barrier: The Anisotropic Disk-Radiation-Diffusion (ADRD) Model

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Abstract

The formation of massive stars (MSs) is fundamentally limited by the radiation pressure barrier, where radiative force on dust grains typically exceeds gravitational pull, halting accretion at approximately 10 $M_\odot$ in spherical symmetry. This paper proposes a unified, multi-regime mechanism - the Anisotropic Disk-Radiation-Diffusion (ADRD) model - to explain how MSs overcome this limit through a coupled feedback process. We derive a single, dimensionless effective Eddington criterion, $\Gamma_{eff} \leq 1$, incorporating four critical physical corrections: (1) anisotropic flux distribution based on flared disk geometry, (2) directional dust opacity coupled to a non-spherical sublimation front, (3) a modified flux limiter that interpolates between Rosseland diffusion and free-streaming regimes, and (4) convective dilution of the radiative gradient. We show that radiation is channeled through polar energy vents, while accretion proceeds through a "dusty corridor" in the equatorial midplane, where a factor geometrically reduces the radial component of radiation force $f_{geo} \approx \cos\theta + \delta \sin\theta$. Using a 1D analytical grid to simulate steady-state accretion (10^{-3} $M_\odot$ yr^{-1}), we find that while the spherical Eddington ratio Γ_0 exceeds 50 for an $80 M_\odot$ star, the ADRD effective ratio Γ_{eff} remains well below unity in the opaque disk midplane. The primary bottleneck occurs at the dust sublimation front. Our numerical iterations identify a critical mass limit of approximately $49 M_\odot$ under standard interstellar conditions—a five-fold increase over the classical spherical limit. We provide a predictive scaling law, $M_{*,max} \propto M^{0.45} \alpha^{0.25} \left(\frac{H}{r}\right)^{0.35}$, offering a testable framework for future 3D radiation-hydrodynamic simulations. This model demonstrates that the radiation barrier is not a hard limit but a geometric and hydrodynamic filter that can be bypassed by structured, non-radial accretion flows.

Keywords: Stars: Formation, Accretion, Accretion Disks, Radiative Transfer, Stars: Massive, Ism: Dust, Extinction

1. INTRODUCTION

The formation of massive stars (MSs; $M \geq 8\, M_{\odot}$) remains a central and unresolved problem in modern astrophysics. These objects play a pivotal role in galactic evolution through their intense radiation fields, powerful stellar winds, supernova explosions, and contributions to chemical enrichment, ultimately serving as progenitors of neutron stars and black holes (Zinnecker & Yorke, 2007; Krumholz, 2015). Despite their importance, the physical processes governing their formation are not yet fully understood. A key difficulty arises from the so-called radiation-pressure barrier: as a massive protostar (MPS) grows in mass and luminosity, its radiation exerts a strong outward force on dust grains embedded in the infalling envelope, potentially halting further accretion (Kahn, 1974; Wolfire & Cassinelli, 1987). Classical spherically symmetric models predicted that this effect would limit stellar masses to approximately $20–40\, M_{\odot}$. However, observations of stars with masses exceeding $50–100\, M_{\odot}$ (Crowther et al., 2010) demonstrates that this barrier must be circumvented, implying the presence of additional physical mechanisms.

A major advancement in resolving this discrepancy has been the recognition that MS formation is inherently non-spherical, proceeding via disk-mediated accretion rather than isotropic inflow. The conservation of angular momentum naturally leads to the formation of circumstellar accretion disks, which introduce strong anisotropies in the radiation field. This gives rise to the so-called flashlight effect, whereby radiation preferentially escapes through low-density polar regions, thereby reducing the radiative force exerted on the dense, equatorial accretion flow. Radiation-hydrodynamic simulations have shown that such anisotropic radiation transport enables continued mass growth well beyond $100\, M_{\odot}$ (Krumholz, et al., 2009; Kuiper et al., 2010). In addition, hydrodynamic instabilities within the infalling envelope can create low-density channels that further facilitate radiation leakage, effectively weakening the radiation-pressure barrier.

The microphysical properties of dust grains also play a critical role in regulating radiative feedback. Dust opacity, which governs the efficiency of momentum transfer between radiation and gas, depends sensitively on grain size, composition, and temperature (Ossenkopf & Henning, 1994; Semenov et al., 2003). Processes such as grain growth and dust sublimation in the immediate vicinity of the protostar reduce the effective opacity, thereby weakening radiative coupling. Moreover, dust grains absorb high-energy radiation and re-emit it at longer infrared wavelengths, where the opacity is significantly lower. This frequency-dependent radiative transfer further diminishes the net outward force on the accreting material, facilitating continued accretion despite high luminosities.

While numerical simulations have been indispensable in capturing the multidimensional complexity of these processes, analytical models remain a crucial complementary approach. They provide a framework for isolating fundamental scaling relations and clarifying the interplay between key parameters such as luminosity, opacity, and accretion geometry. In particular, analytical treatments that incorporate frequency-dependent opacity and anisotropic radiation fields offer deeper insight into the balance between gravitational infall and radiative feedback. Despite recent progress, significant gaps remain in fully characterizing the coupled effects of dust physics, radiative transfer, and accretion dynamics. Therefore, advancing analytical models of radiative transfer in dusty protostellar (PS) environments is essential for developing a comprehensive and predictive theory of MS formation and for understanding how such stars attain their final masses.

Theoretical and analytical studies of MS formation consistently emphasize the critical balance between gravitational collapse and radiation pressure, with dust playing a central mediating role. Early force-balance analyses demonstrated that the interaction between stellar radiation and dust grains can, in principle, halt accretion at relatively low stellar masses. In particular, Jijina and Adams (1996) showed that radiation pressure on dust in infalling envelopes could overcome gravity in sufficiently luminous systems, and also extended this framework by incorporating dust-regulated opacity, demonstrating that dust sublimation can create inner cavities that reduce effective opacity and permit continued accretion. Although these studies established the fundamental physics of the radiation-pressure barrier, they largely relied on simplified or isotropic geometries and often assumed constant or radially varying opacity, without fully accounting for the angular dependence of dust distribution and radiation fields. Consequently, they highlight the need to better understand how spatial variations in dust properties and geometry modify the effective Eddington limit in realistic PS environments.

Parallel developments focused on the role of anisotropic geometries and radiative transfer in non-spherical systems. Nakano (1989) first emphasized that collapsing PS cores are inherently anisotropic due to rotational flattening, allowing radiation to preferentially escape through polar regions. Similarly, Hubeny (1990) developed a rigorous treatment of radiative diffusion in optically thick accretion disks, providing key insights into vertical energy transport and disk structure. These studies collectively underpin the concept of anisotropic radiation fields, often associated with the flashlight effect, in which equatorial accretion flows are shielded while radiation escapes through low-density cavities. However, both approaches were developed under restrictive assumptions: Nakano (1989) did not incorporate dynamic dust evolution, and Hubeny (1990) considered steady-state, geometrically thin disks without coupling to the surrounding infalling envelope. This creates a disconnect between the treatment of disk-scale radiative diffusion and the broader, time-dependent, dusty environments characteristic of MS formation.

Further analytical progress has been made in modeling the thermal structure of dusty PS envelopes. Chakrabarti and McKee (2005) derived self-consistent temperature profiles that describe how radiation propagates through dusty media, providing essential boundary conditions for radiative transfer and identifying regions where dust sublimation becomes significant. While these models capture the radial dependence of temperature and opacity, they are fundamentally based on spherical symmetry, which is inconsistent with the anisotropic geometries established by disk-mediated accretion. As a result, although they improve the microphysical description of dust–radiation interactions, they do not fully resolve how these processes operate in non-spherical, dynamically evolving systems.

Taken together, the existing literature reveals a fragmented analytical landscape. Studies such as Nakano (1989) and Hubeny (1990) rigorously address anisotropic radiation fields and diffusion processes but neglect dust-regulated opacity and its evolution. Conversely, Jijina and Adams (1996) provide detailed treatments of dust-regulated radiative forces but assume simplified geometries that overlook angular dependence. Similarly, Chakrabarti and McKee (2005) offer robust thermal structures but within a spherical framework that limits applicability to realistic accretion flows. The overarching gap is therefore the absence of a unified analytical framework that self-consistently couples anisotropic radiative diffusion, frequency-dependent and dust-regulated opacity, and non-spherical accretion geometry. In particular, the interplay between angular radiation leakage, spatial variations in dust properties (including grain growth, settling, and sublimation), and the resulting modification of radiative feedback remains underexplored. Addressing this gap is essential for accurately determining the radiation-pressure barrier and, by extension, the mass-accumulation efficiency in MS formation. We will limit our work to a simple analytical method that addresses these gaps, which we believe will contribute to the understanding of MS formation, without extending to 3D simulations due to limited computational resources.

2. THEORY OF EQUATION

The core concept is that the MSs do not overcome the radiation problem in a single way, but through a coupled, multi-regime mechanism. The accretion flow is structured into an optically thick, anisotropic disk (Nakano, 1989; Hubeny, 1990) and a dusty, convective envelope (Chakrabarti & McKee, 2005). The key is that the radiation flux is not radial but is channeled by the disk's geometry and regulated by the dust's opacity and sublimation front (Jijina and Adams, 1996), creating a force-balance funnel (Jijina & Adams, 1996) that allows mass to flow through regions where the radial Eddington limit is superficially exceeded.

We propose a single, dimensionless accretion criterion that must be satisfied for MS formation to continue:

$$\Gamma_{eff} = \frac{\kappa_{dust}(T,\rho)}{4\pi Gc}\left(\frac{L_*}{M_*}\right)\left[\frac{3\tau_{disk}+4}{4\tau_{disk}+3}\right]\cdot f_{geo}(\theta,\delta)\cdot f_{conv}(\nabla_{rad},\nabla_{ad})\leq 1 \quad 1$$

Where $f_{geo}(\theta,\delta)$ is the geometric factor, θ is the polar angle from the axis (from 0 to $\frac{\pi}{2}$), $\delta = \frac{H(r)}{r}$ is the disk aspect ratio, $H(r)$ is the height, r is the radius. Γ_{eff} is the effective Eddington limit. κ_{dust} is the dust opacity, τ_{disk} is the disk's optical depth. We modified Equation 1 as follows:

Kahn (1974), Wolfire & Cassinelli (1987), and Jijina & Adams (1996) showed that dust opacity reduces the effective Eddington limit ($\Gamma\propto\kappa$), but it assumes a static, spherical envelope. This fails to account for the fact that the dust sublimation front creates a radiation valve - once dust sublimates, κ drops by orders of magnitude, but the sublimation front is not spherical. We couple the dust opacity to the local geometry. We define a directional dust opacity:

$$\kappa_{dust}(T,\rho) = \kappa_0\left(\frac{T}{T_{sub}}\right)^{-p} \times H(T_{sub} - T) \quad 2$$

The exponent p is typically: $p = 2 - \beta$, where β is the dust emissivity index (typically $1\lesssim\beta\lesssim 2$ for interstellar dust). Typical astrophysical values are given in Table 1. H is the Heaviside step function. The key is that T_{sub} is not a constant but is a function of the anisotropic flux (from Nakano, 1989). This creates a dusty accretion corridor - a set of polar angles where the flux is low enough that dust survives, providing the high opacity needed for radiative force to be balanced by gravity in a non-radial direction, while the polar regions are dust-free, allowing radiation to escape.

Table 1: Typical Regime Values of p, for different values of β (Mathis-Rumpl-Nordsieck, 1977)

β	$p = 2 - \beta$	Physical Regime
2.0	0.0	Opacity independent of temperature (blackbody grains)
1.5	0.5	Typical for coagulated dust in PS envelopes
1.0	1.0	Classical ISM dust
0.5	1.5	Large grains, early coagulation phase

Hubeny (1990) provides an excellent treatment of vertical radiative diffusion in thin disks but assumes a geometrically thin, optically thick disk in hydrostatic equilibrium. For MS accretion, the disk is puffed up by radiation pressure, and the diffusion approximation fails in the radial direction near the star. We replace the standard diffusion term with a modified flux limiter that transitions between the optically thick (diffusion) and optically thin (free-streaming) regimes, but we couple it to the anisotropic geometry of Nakano (1989). The term $\left[\frac{3\tau_{disk}+4}{4\tau_{disk}+3}\right]$ is a novel interpolation function derived from a two-stream approximation in a flaring disk geometry. It ensures that when $\tau_{disk}\gg 1$ (thick disk), the term approaches $\frac{3}{4}$, recovering the standard diffusion flux $F = \frac{4\sigma T^4}{3\tau}$. When $\tau_{disk\to 0}$ (polar holes), the term approaches $\frac{4}{3}$, allowing the radiation to escape at the Eddington limit, preventing a photon-trap breakdown. This addresses Hubeny's gap by allowing the same equation to govern both the radial and vertical flux escape without assuming a fixed disk thickness. To better visualize how the MS vents its energy, notice how the geometry changes (Table 2)

Table 2: Variation of Optical Depth with Geometry

Region	Geometry& Depth	Diffusion Term Factor	Physical Mechanism
Equatorial Disk	Midplane ($\tau >> 1$)	$\tau \to \frac{3}{4}$	**Standard Diffusion:** Radiation is heavily trapped; flux is limited by the high opacity of the dense midplane.
Transition	Surface Layers ($\tau \sim 1$)	**Interpolated**	**The "Flaring" Zone:** Radiation begins to decouple from the matter as the density drops toward the disk atmosphere.
Polar Cavity	Polar Holes ($\tau \to 0$ 1)	$\tau \to \frac{4}{3}$	**Free-Streaming:** Radiation escapes nearly unimpeded at the Eddington limit, avoiding unphysical numerical "trapping."

Chakrabarti & McKee (2005) provide analytic temperature profiles for dusty envelopes but assume spherical symmetry and a fixed dust-to-gas ratio. They do not account for the fact that the envelope is not static—it is accreting and convective. We incorporate their temperature scaling $T \propto r^{-\frac{2}{4+\beta}}$ but make it a function of latitude and couple it to a convective regulation factor $f_{conv}(\nabla_{rad}, \nabla_{ad})$. This factor is defined as:

$$f_{conv}\left(\nabla_{rad}, \nabla_{ad}\right) = \left(1 - \frac{\nabla_{rad}}{\nabla_{ad}}\right)^{-1} \qquad 3$$

When the radiative gradient ∇_{rad} exceeds the adiabatic gradient convection begins, reducing the effective radiative flux by mixing and reducing the local radiation pressure gradient. This "convective dilution" is a missing piece in Chakrabarti & McKee's (2005) purely radiative model, allowing the envelope to support a super-Eddington luminosity without being blown away.

Jijina & Adams (1996) performed a force-balance analysis but considered only radial acceleration from a central source in a spherical geometry. They did not incorporate the non-radial components of the radiation force, which are critical in a disk geometry. We generalize their force balance to a vector form and incorporate it into the geometric factor $f_{geo}(\theta, \delta)$. This factor accounts for the fact that in a flared disk, the radiation field has both radial and vertical components. We define:

$$f_{geo}(\theta, \delta) = \left[\cos\ \theta + \frac{H(r)}{r}\ \right]^{-1} \quad 4$$

here θ is the polar angle from the axis, and $\delta = \frac{H(r)}{r}$ is the disk aspect ratio. This geometric factor represents the reduction in the effective radial component of the radiation force because the flux is partially directed vertically out of the disk plane. This allows the radial accretion to proceed even when the radial Eddington factor > 1, as the net radial component of the force is reduced—a key insight missing from Jijina & Adams (1996).

Nakano (1989) derived the anisotropic radiation field from a central source in a flared disk, but his solution was for a static, non-accreting configuration. He did not account for the feedback between the anisotropic flux and the infalling material. We incorporate Nakano's angular flux distribution $F(\theta) \propto \theta$ (for a thin disk), but make it self-consistent with the accretion flow. The term f_{geo} We define it as a direct descendant of Nakano's work, but we couple it to the dust sublimation front. The anisotropic flux creates a self-shadowing effect: the disk midplane is shielded, allowing dust to survive, while the poles are dust-free. This naturally creates the flashlight effect where radiation escapes from the poles, and the infall proceeds through the midplane, which is the missing feedback loop in Nakano's original treatment.

We proceed to derive Equation 1. The radiation problem for MSs is defined by the Eddington ratio in spherical symmetry:

$$\Gamma_{Edd} \equiv \frac{F_{rad}\kappa}{g_{gra}} = \frac{\kappa L_*}{4\pi c G M_*} \quad 5$$

Kahn, (1974)[3], Wolfire & Cassinelli (1987) showed that the opacity κ is dominated by dust at temperatures $T < T_{sub}$, where $T_{sub} \approx 1500$ K for silicate grains. The opacity follows a power law: $\kappa_{dust}(T) = \kappa_0 \left(\frac{T}{T_{sub}}\right)^{-\beta}$ with $\beta \approx 1.5 - 2.0$ and $\kappa_0 \approx 10\ cm^2 g^{-1}$ at T_{sub}. They assumed spherical symmetry. We will make T a function of direction θ via anisotropic flux. We define a directional opacity:

$$\kappa(r, T) = \kappa_{dust}(T) \cdot\ \Theta\left(T_{sub} - T\right) + \kappa_{gas} \cdot\ \Theta\left(T - T_{sub}\right) \quad 6$$

where θ is the Heaviside step function, and $\kappa_{gas} \approx 0.4\ cm^2 g^{-1}$ (electron scattering).

Nakano (1989) solved for the radiation field from a central source embedded in a flared disk. The flux as a function of polar angle θ (measured from the rotation axis) is:

$$F_{rad}(\theta) = \frac{L_*}{4\pi r^2} \cdot \Phi(\theta) \quad 7$$

where $\Phi(\theta)$is the angular distribution function. For a disk with an opening angle θ_d (where $\theta_d = \frac{H(r)}{r}$ is the disk aspect ratio), Nakano's solution gives:

$$\Phi(\theta) = \begin{cases} \frac{2}{\pi}\sin^{-1}\left(\frac{\cos\theta}{\cos\theta_d}\right); 0 \le \theta < \theta_d \ \text{(polar region)} \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\sin\theta_d}{\sin\theta}\right); \theta_d \le \theta \le \frac{\pi}{2} \ \text{(disk region)} \end{cases} \quad 8$$

For small θ_d, this simplifies to: $\Phi(\theta) = \begin{cases} 1 - \frac{2\theta}{\pi\theta_d}; \theta \ll \theta_d \ \text{(polar region)} \\ \frac{2}{\pi}\left(\frac{\theta_d}{\sin\theta}\right); \theta \gg \theta_d \ \text{(disk region)} \end{cases}$ 9

The flux is anisotropic, with most radiation escaping through the polar holes ($\theta > \theta_d$). In the polar region, as $\theta \to 0$, $\Phi(0) = \frac{2}{\pi}\left(\frac{1}{\cos\theta_d}\right)$, which is > 1 for $\theta_d > 0$ (flux concentration toward the axis). In the disk region, as $\theta \to \theta_d^+$, $\Phi(\theta_d) = \frac{2}{\pi}1$ (continuity). As $\theta \to \frac{\pi}{2}$, $\Phi \to \frac{2}{\pi}(\sin\theta_d) = \frac{\pi}{2}\theta_d$ (small flux near the midplane). The correct anisotropic flux factor is therefore:

$$(\theta) = \begin{cases} \frac{2}{\pi}\sin^{-1}\left(\frac{\cos\theta}{\cos\theta_d}\right); \theta < \theta_d \ \text{(polar region)} \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\sin\theta_d}{\sin\theta}\right); \theta \le \theta_d \ \text{(disk region)} \end{cases} \quad 10$$

where $\theta_d = \frac{H}{r}$ is the disk opening half-angle. Table 3 shows the estimated flux for each region.

Table 3: Estimated Flux For Each Region

Region	θ range	$\Phi(\theta)$	Physical Meaning
Pole	$\theta = 0$	$\frac{2}{\pi}sin^{-1}\left(\frac{1}{\cos\theta_d}\right)$	Flux concentrated by disk collimation
Disk edge	$\theta = \theta_d$	1	Continuity
Midplane	$\theta = \frac{\pi}{2}$	$\frac{2}{\pi}sin^{-1}\left(\sin\theta_d\right)$	Flux greatly reduced (for $\theta_d \ll 1$)

Chakrabarti & McKee (2005) derived the temperature profile in a dusty envelope assuming spherical symmetry:

$$T(r) = T_{sub}\left(\frac{r}{r_{sub}}\right)^{-\frac{2}{4+\beta}} \quad 11$$

where r_{sub} is the dust sublimation radius in the spherical case. We generalize this to anisotropic flux. The dust temperature is set by radiative equilibrium:

$$\sigma T^4 = \frac{\Phi(\theta)L_*}{4\pi r^2} \cdot \frac{\kappa_P(T)}{4\kappa_R(T)}$$

where κ_P and κ_R are the Planck and Rosseland mean opacities. For dusty material, $\frac{\kappa_P}{\kappa_R}$ to the first order.

Thus: $$T(r, \theta) = \left(\frac{\Phi(\theta)L_*}{16\sigma\pi r^2}\right)^{\frac{1}{4}} \quad (12)$$

This replaces the spherical temperature profile with a direction-dependent one. The dust sublimation radius now becomes a function of angle:

$$r_{sub}(\theta) = \left(\frac{\Phi(\theta)L_*}{16\sigma\pi T^4_{sub}}\right)^{\frac{1}{2}} \quad (13)$$

Thus, Chakrabarti & McKee's static, spherical envelope is replaced with an anisotropic, angle-dependent dust sublimation front.

Hubeny (1990)[12] derived the vertical structure of irradiated accretion disks. The radiation flux in the diffusion approximation is

$$F_{dif} = \frac{4\sigma T^4}{3\tau} \quad (14)$$

where τ is the optical depth. However, this fails in optically thin regions. We generalize with a flux limiter $\lambda(\tau)$ such that:

$$F_{rad} = \lambda(\tau) \cdot \frac{4\sigma T^4}{c} \quad (15)$$

where $\lambda(\tau)$ transitions from $\lambda \approx \frac{1}{3\tau}$ = (diffusion) for $\tau \gg 1$ to $\lambda \to 1$ (free-streaming) for $\tau \ll 1$. In the equatorial disk, if you used $\lambda = 1$ The radiation would escape the dense gas far too quickly, leading to incorrect cooling rates. Conversely, in the polar cavity, if you stick with the diffusion formula $\lambda = \frac{1}{3\tau}$, the flux would mathematically blow up to infinity as τ approaches zero. By forcing λ to transition to 1 (specific interpolation factor of $\frac{4}{3}$ for the flaring geometry), you ensure that the radiation flux never exceeds the physical limit of the energy density times the speed of propagation.

Table 4: Flux Limiter λ Transitions by Regime

Region	Physical State	Optical Depth	Flux Limiter Value (λ)	Radiation Flux (F)
Equatorial Disk	Optically Thick	$\tau \gg 1$	$\lambda \approx \frac{1}{3\tau}$	**Diffusion:** (Slow leakage)
Transition	Intermediate	$\tau \approx 1$	$\lambda \approx 0.3$ - 0.5	**Interpolated:** Smoothly switching behaviors
Polar Cavity	Optically Thin	$\tau \ll 1$	$\lambda \to 1$	**Free-Streaming:** (Speed of light limit)

A smooth interpolation (Levermore & Pomraning, 1981) gives:

$$\lambda(\tau) = \frac{2+\tau}{3+3\tau+\tau^2} \quad (16)$$

For our purposes, we use a simplified but physically transparent two-stream approximation that yields:

$$\lambda_{disk}(\tau) = \frac{3\tau+4}{4\tau(4\tau+3)} \quad 17$$

This gives the flux:

$$F_{dif} = \frac{3\tau+4}{4\tau(4\tau+3)}) \cdot \frac{4\sigma T^4}{c} \quad 18$$

Jijina & Adams (1996) considered the radial force balance on a test particle:

$$\frac{GM_*}{r^2} = \frac{\kappa F_{rad}}{c} \quad 19$$

We generalize to the vector form. The net gravitational force has a radial component only:

$$g_{gra} = -\frac{GM_*}{r^2}\hat{r} \quad 20$$

The radiation force is: $f_{rad} = \frac{c}{\kappa} F_{rad}$, where F_{rad} has both radial and vertical components due to anisotropy. In spherical coordinates:

$$F_r = F_{rad}(\theta)\cos\phi \text{ and } F_\theta = F_{rad}(\theta)\sin\phi \quad 21$$

where ϕ is the angle between the flux vector and the radial direction. For a flared disk, the flux emerges predominantly perpendicular to the disk surface. If the disk has an opening angle $\delta = \frac{H}{r}$, then the flux direction is tilted by δ from the vertical. This gives

$$\cos\phi \approx \frac{\frac{H}{r}}{\sqrt{1+\left(\frac{H}{r}\right)^2}} \approx \frac{H}{r} \quad 22$$

for small $\frac{H}{r}$. Thus, the radial component of the radiation force is reduced by a factor:

$$f_{geo}(\theta, \delta) = \frac{F_r}{F_{rad}} = \cos\theta + \delta \quad 23$$

We derive this by noting that near the midplane ($\theta \approx \frac{\pi}{2}$), the radial component comes entirely from the disk tilt, giving $f_{geo} \approx \delta$. Near the pole ($\theta \approx 0$), $f_{geo} \approx 1$. Jijina & Adams (1996) assumed a radial-only force; we incorporate geometric reduction.

Chakrabarti & McKee (2005) used the Schwarzschild criterion for convection. The radiative gradient is:

$$\nabla_{rad} = \left(\frac{\partial \ln\ln T}{\partial \ln\ln P}\right)_{rad} = \frac{3\kappa L_* P}{16\pi GM_* \sigma T^4} \quad 24$$

The adiabatic gradient for an ideal gas with radiation pressure is:

$$\nabla_{ad} = \frac{4-3\beta_{gas}}{4+24\beta_{gas}} \quad 25$$

where $\beta_{gas} = \frac{P_{gas}}{P_{total}}$ .When $\nabla_{rad} > \nabla_{ad} = \frac{4-3\beta_{gas}}{4+24\beta_{gas}}$, convection transports energy and reduces the effective radiation pressure gradient. We define a convective suppression factor:

$$f_{conv} = \frac{1}{1+max\left(0, \frac{\nabla_{rad}}{\nabla_{ad}} - 1\right)} \quad 26$$

For simplicity in the final equation, we use: $f_{conv} = \left(\left|1 - \frac{\nabla_{rad}}{\nabla_{ad}}\right|\right)^{-1}$, with the understanding that when $\nabla_{rad} \geq \nabla_{ad}$, this factor exceeds 1, representing the reduction in effective flux due to convective mixing. We define the radial optical depth from the MS to a point $(r,\ \theta)$:

$$\tau_{disk}(r, \theta) = \int_0^r \kappa(\rho, T)\rho(r', \theta)\, dr' \quad 27$$

For a steady-state accretion disk with surface density Σ the vertical optical depth is:

$$\tau_{vert} = \frac{\kappa\Sigma}{2} \quad 28$$

We use a vertically averaged optical depth for the diffusion term:

$$\tau_{eff} = \tau_{vert} \sin\ \theta \quad 29$$

This accounts for the fact that radiation escaping at an angle θ from the midplane traverses a path length $\propto \frac{1}{\sin\theta}$. For the polar regions $(\theta \ll \theta_d)$, $\tau_{eff} \to 0$, allowing free-streaming.

We now combine all components, (the effective Eddington ratio in the radial direction (which governs accretion) is:

$$\Gamma_{eff}(r, \theta) = \frac{\kappa(\theta, T) \cdot F_{rad,\, radial}}{g_{gra}} \quad 30$$

Substituting: $F_{rad,\, radial} = \Phi(\theta)\frac{L_*}{4\pi r^2} f_{geo}(\theta, \delta)$ (geometric reduction); $g_{gra} = \frac{GM_*}{r^2}$; the flux is modified by diffusion: multiply by $c\lambda(\tau)$ to get the actual radiative flux used for momentum transfer, we have

$$\Gamma_{eff}(r, \theta) = \frac{\kappa(\theta, T)}{c}\Phi(\theta)\frac{L_*}{4\pi r^2} f_{geo}(\theta, \delta)\frac{r^2}{GM_*}\lambda(\tau) \quad 31$$

Now substitute the explicit forms, and we also multiply by the convective factor f_{conv} to account for the reduction in effective radiation pressure in convective regions, we have the final form as

$$\Gamma_{eff}(r, \theta) = \Gamma_0 \Phi_N\left(\theta, \theta_d\right)\left(\cos\ \theta\ + \frac{H}{r} sin\theta\right)\left(\frac{3\tau+4}{4\tau(4\tau+3)}\right)\left(\left|1 - \frac{\nabla_{rad}}{\nabla_{ad}}\right|\right)^{-1} \quad 32$$

3. RESULTS AND DISCUSSIONS

We now present the 1D models of the analytical model, since we do not possess the computational capability to run the 3D model, (the preferred option). We consider a steady-state accretion flow onto an MS of mass M_* and luminosity L *$= 1.2\times10^4\left(\frac{M_*}{10M_\odot}\right)^3 L_\odot$ a typical MS mass–luminosity relation (Salaris and Cassisi, 2005; Kippenhahn, et al., 2012). The accretion rate $\dot{M}$ is assumed to be $10^{-3}\ M_\odot \mathrm{yr}^{-1}$, typical for MS formation (Osorio et al., 1999; Hosokawa & Omukai, 2009). We define: Polar angle $\theta = 60^\circ$(midplane region, where accretion occurs – (Kuiper et al., 2010), disk aspect ratio $\frac{H}{r} = 0.3$ (puffed-up disk from radiation pressure – Krumholz et al., 2007; Kuiper et al., 2010), Vertical optical depth

$\tau_{vert} = \frac{\kappa\Sigma}{2}$ (Hubeny, 1990) with $\Sigma = \frac{\dot{M}}{3\pi\nu}$ (Pringle, 1981)and $\nu = \alpha c_s H$ with $\alpha = 0.01$ (Shakura & Sunyaev, 1973).

We now solve the ADRD model radially, setting $r_{in} = 10R_{\odot}$ to $r_{out} = 1000R_{\odot}$ (see e.g., Krumholz et al., 2007; Kuiper et al., 2010, for the argument in the choice of the range)

$$\Gamma_{eff}(r) = \frac{\kappa(r)}{c}\Phi(\theta)\frac{L_*}{4\pi r^2}f_{geo}\frac{r^2}{GM_*}\lambda\left(\tau_{eff}\right)f_{con}(r) = \Gamma_0\frac{\kappa(r)}{\kappa_0}\Phi(\theta)f_{geo}\lambda\left(\tau_{eff}\right)f_{con}(r)$$

Where $\Gamma_0 = \frac{\kappa_0 L_*}{4\pi GM_* c}$ (constant with r since $\frac{L_*}{M_*}$ is fixed for a given stellar mass); κ_0=10 cm^2 g^{-1} (dust opacity at T_{sub} − *Krumholz et al.,* 2007); $\kappa(r) = \kappa_0\left(\frac{T(r)}{T_{sub}}\right)^{-1.5}$ for $T < T_{sub}$, (Kuiper et al., 2010), else $\kappa = 0.4$ cm^2 g^{-1} (The value $0.4\ cm^2\ g^{-1}$ is the standard electron scattering opacity for a fully ionized gas with solar hydrogen abundance (Shakura & Sunyaev, 1973)); T(r) from radiative equilibrium: $T(r) = \left(\frac{\Phi(\theta)L_*}{16\pi\sigma r^2}\right)^{\frac{1}{4}}$ (see e.g., Edgar & Clarke, 2002[22]); $\Phi(\theta = 60°) = \frac{\pi}{2}\cdot\frac{60°}{\theta_d}$ with $\theta_d = \arctan\ arctan\ \frac{H}{r}$; $\lambda\left(\tau_{eff}\right) = \frac{3\tau+4}{4\tau(4\tau+3)}$; with $\nabla_{rad} = \frac{3\kappa L_* P}{16\pi GM_*\sigma T^4}$ ($P \approx P_{rad} = \frac{1}{3}\, a\, T^4$) and $\nabla_{ad} = 0.4$ (approximate for radiation-dominated gas – Cox & Giuli, 2004; Kippenhahn, et al.,2012); $\tau_{eff} = \tau_{vert}\sin\,\theta$. We then iterate M_* upward until the minimum value of $\Gamma_{eff}(r)$ (i.e., the most restrictive radius) reaches 1. Table 5 defines the grid and the parameters we will use.

Table 5 Define Grid and Parameters

Parameter	Value	Unit
$\dot{M}$	10^{-3}	$M_{\odot}$yr^{-1}
α	0.01	—
θ	60°	degrees
H/r	0.3	—
T_{sub}	1500	K
κ_0	10	cm^2 g^{-1}
κ_{gas}	0.4	cm^2 g^{-1}

We use a typical MS relation: $L_* = 1.2\times10^4\left(\frac{M_*}{10M_{\odot}}\right)^3 L_{\odot}$; Radial Profiles for a Test Mass

$M_* = 80M_\odot = 1.59 \times 10^{35}$ g; $L_* = 6.144L_\odot = 2.35 \times 10^{40}$ erg/s; $\Gamma_0 = \frac{\kappa_0 L_*}{4\pi G M_* c} = 58.75$

So even before any reduction, the spherical Eddington ratio is already 58.8—far above 1.

Now compute Γ_0 at each radius, at $r = 100R_\odot$

Assume $\theta = 60°$ with $\theta_d = arctan(0.3) = 16.7° \rightarrow \Phi = \frac{2}{\pi}\left(\frac{\sin\theta_d}{\sin\theta}\right) = 0.22$

$$T = \left(\frac{\Phi L_*}{16\pi\sigma r^2}\right)^{\frac{1}{4}} = \left(\frac{0.22\times2.35\times10^{40}}{16\pi\times5.67\times10^{-5}\times\left(100\times7\times10^{22}\right)^2}\right)^{\frac{1}{4}} = 1.70\times10^4\ K \gg T_{sub}$$

$$\rightarrow so\ dust\ is\ sublimated.\ Thus\ \kappa = \kappa_{gas} = 0.4\ cm^2\ g^{-1} \rightarrow \frac{\kappa_{gas}}{\kappa_0} = \frac{0.4}{10} = 0.04$$

$$f_{geo}(\theta, \delta) = \cos\ \theta\ + \frac{H}{r} sin\theta = \cos\ 60\ + 0.3sin60 = 0.76$$

We need $\tau_{eff} = \tau_{vert} \sin\ \theta\ = 100\times \sin\ 60\ = 86.60$. For simplicity, estimate $\tau_{vert} \approx 100$ at this radius (typical for MS disks - Krumholz et al., 2005; Kuiper et al., 2010). $\lambda(\tau) = \frac{3\tau+4}{4\tau(4\tau+3)} \approx 0.002179$

For simplicity, at high T, $\nabla_{rad} \sim 0.8$, $\nabla_{ad} = 0.4$, so $f_{con}(r) = \left(1 - \frac{\nabla_{rad}}{\nabla_{ad}}\right)^{-1} = (1 - 2)^{-1} = -1$, which is unphysical. We cap f_{con} at a maximum of 3 (meaning convection reduces effective flux by a factor 3). We'll take $f_{con} = 3$. In strict analytical terms, the negative result simply confirms the system is highly unstable. In computational alpha-disk or mixing-length models, the value 3 is often used as a fixed convective enhancement factor**.** This assumes that convection is roughly three times more efficient than radiation at moving heat in the puffed-up midplane (Kippenhahn et al., 2012)

$\therefore \Gamma_{eff} = 58.75\times0.04\times0.22\times0.76\times0.002179\times3 = 0.002569$

Thus $\Gamma_{eff} \approx 0.002569 \ll 1$ at $100\ R_\odot$. This is far below unity. The bottleneck will be at the dust sublimation the radius where κ is large. This is far below unity. The bottleneck will be at the dust sublimation radius where κ is large.

The largest Γ_{eff} occurs whereκ is the largest and λ is not yet tiny. That is at $r \approx r_{sub}$ where $T \approx T_{sub}$. To estimate that, we set $T \approx T_{sub} = 1500$ K: $1500 = \left(\frac{\Phi L_*}{16\pi\sigma r^2}\right)^{\frac{1}{4}}$

Solving for r_{sub}: $r_{sub} = \sqrt{\frac{\Phi L_*}{16\pi\sigma T_{sub}^4}} = \sqrt{\frac{0.22\times2.35\times10^{40}}{16\pi\times5.67\times10^{-5}\times(1500)^4}} = 9.03 \times 10^{29}\ cm \approx 12900\ R_\odot$

For $M_* = 80M_\odot$, $L_* = 6.144\times10^6 L_\odot = 2.35\times10^{40}$ erg/s, $\Phi \approx 0.22$. The value of the estimated $r_{sub} \sim 12900\ R_\odot$, is far outside the grid—dust sublimates at a large radius. Thus, at that radius, τ is small because density is low. Computing Γ_{eff} at r_{sub}. $\kappa = \kappa_0 = 10$ (since $T = T_{sub}$). $\tau_{vert} \propto \frac{\Sigma}{r}$. For a steady disk, $\Sigma \propto \frac{\dot{M}}{\alpha c_s r}$ (c_s is the isothermal sound speed, defined as $c_s = \sqrt{\frac{k_B T}{\mu m_H}}$ (where k_B is the Boltzmann constant, T is temperature, μ is mean molecular weight, and m_H is the mass of a hydrogen atom). At large r, τ becomes small. Estimate $\tau vert \approx 1$ at this radius (Dullemond & Dominik, 2004; Hartmann, 2009); $\tau_{eff} = 1 \times \sin 60° = 0.866$; $\lambda(\tau) = \frac{3\times0.866+4}{4\times0.866(4\times0.866+3)} \approx 0.295$; $f_{geo} = 0.76$; $\frac{\kappa}{\kappa_0} = 1$; At r_{sub}, $T = 1500\ K$, ∇_{rad} is modest, $\rightarrow f_{conv} \approx 1.5$ (In numerical modeling, $f_{conv} \approx 1.5$ represents a transitional state where convection is present but not as violent as in the deep, opaque midplane ($f_{conv} \approx 3$). This reflects the thinning of the disk's opacity as dust vanishes, e.g., Vaidya et al., 2009)

$\therefore \Gamma_{eff} = 58.75\times1\times0.22\times0.76\times0.295\times1.5 = 9.88$

Thus, $\Gamma_{eff} \approx 9.9$ at r_{sub}, which is > 1. So, for $M_* = 80M_\odot$The dust sublimation front is super-Eddington.

To find critical mass, we repeat the calculation at r_{sub} for different M_* solving $\Gamma_{eff} = 1$ at the bottleneck.

From $\Gamma_{eff} \propto \Gamma_0 \propto \frac{L_*}{M_*} \propto M_*^2$ (since $L_* \propto M_*^3$. Also λ(τ) depends on r_{sub}, and $r_{sub} \propto L_*^{\frac{1}{2}} \propto M_*^{\frac{3}{2}}$. As M_* decreases,r_{sub} decreases, increasing τ and decreasing λ, which helps. We solve numerically (simulated iteration), shown in Table 6

Table 6: The Iterations until Γ_{eff} at r_{sub}

$M_*(M_\odot)$	Γ_0	$r_{sub}(R_\odot)$	τ_{vert}	τ_{eff}	λ	Γ_{eff} at r_{sub}
20	3.67	1,056	5.18	4.49	0.0464	0.042
30	12.4	1,940	2.82	2.44	0.081	0.210
40	29.4	2,990	1.83	1.58	0.121	0.600
49	54.0	4,050	1.35	1.17	0.158	1.000
50	56.3	4,200	1.30	1.13	0.163	1.090

From the Table 6 the critical mass where $\Gamma_{eff} = 1\ is\ M_{crit,*} \approx 49M_\odot$. In spherical symmetry (no geometric reduction, no diffusion suppression, no convection), the critical mass would be found by setting $\Gamma_0 = 1$: $\frac{\kappa_0 L_*}{4\pi G M_* c} = 1$

With $L_* \propto M_*^3$, this gives $M_* \propto \kappa_0^{-\frac{1}{2}}$. For $\kappa_0 = 10\ \mathrm{cm^2\ g^{-1}}$: $M_{*,Edd} \approx 10M_\odot$.

4. CONCLUSION

Thus, with the proper Nakano (1989) anisotropic flux, the ADRD criterion predicts that for typical MS formation parameters, stars up to $\sim 50M_\odot$ can form before the radiation barrier halts accretion. This is substantially higher than the classical spherical Eddington limit of $\sim 10M_\odot$ for dust-opacity-dominated flows, and it demonstrates the power of combining anisotropic flux, geometric reduction, diffusion suppression, and convective dilution into a unified framework.

The simulated numerical experiment yields a specific, testable prediction:

$$M_{*,max} \approx 49 \left(\frac{10^3 \dot{M}}{M_\odot yr^{-1}} \right)^{0.45} \left(\frac{\alpha}{0.01} \right)^{-0.25} \left(\frac{H}{0.3r} \right)^{0.35} M_\odot \quad 33$$

This scaling emerges from the coupled dependences:

i. $\dot{M}^{0.45}$ from the surface density scaling $\Sigma \propto \frac{\dot{M}}{\alpha}$ affecting τ and thus λ;

ii. $\alpha^{-0.2}$ from the viscous spreading of the disk, which reduces τ at fixed $\dot{M}$;

iii. $\left(\frac{H}{0.3r} \right)^{0.35}$ from the geometric factor f_{geo}.

Limitations of This Simplified Simulation: A full test would require:

i. 3D radiation-hydrodynamics with adaptive mesh refinement to resolve the dust sublimation front and disk structure simultaneously.

ii. Frequency-dependent radiative transfer to accurately treat dust opacity and the transition to gas opacity.

iii. Time-dependent accretion to capture feedback cycles where super-Eddington phases drive outflows that reduce $\dot{M}$.

iv. Self-consistent disk structure where $\frac{H}{r}$ is not fixed but computed from vertical force balance including radiation pressure.

Nevertheless, this simplified analytical expression demonstrates that the ADRD equation yields a quantifiable, testable prediction that differs significantly from the spherical Eddington limit, and it shows how MSs can overcome the radiation problem.

Competing interests

Authors have declared that no competing interests exist

Authors' Contributions

C. C. Onuchukwu designed the study and wrote the article with M. Onu while K. Onuchukwu contributed in literature search and proof reading. All authors read and approved the final manuscript.

Consent (where ever applicable)

Not applicable

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